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ABOUT THIS GUIDE

Edgenuity supplies teachers with a Teacher’s Guide to assist them in helping students succeed in every course. The Teacher’s Guide summarizes both the content that students learn in Algebra II based on the Louisiana Student Standards for Mathematics, and the eight Standards for Mathematical Practice in which students must be engaged for all of this content. The guide organizes the focus standards into units and the learning goals into lessons. Discussion questions for each unit are included, paired with tips for effective discussions to support teachers in hosting them. Online lessons begin with a warm-up, dive into rich instruction, recap what students have learned in a summary, allow students to practice skills, and finally assess students in a quiz. When the structure of the online lessons are used as designed and combined with engaging non-routine problems and deep discussions with the questions provided, teachers are given everything they need to help their students succeed in a blended learning environment.

While technology has changed how content is delivered, it has not removed the student’s need for individualized instruction, remediation, or challenge—support that only a teacher can provide. That’s why each Teacher’s Guide comes with specific instructions for how to use Edgenuity’s innovative course customization tool set. This allows permissioned educators and district administrators to create truly customized courses that can meet the demands of the most rigorous classroom or provide targeted assistance for struggling students.

Finally, the Teacher’s Guide provides helpful resources, including lists of vocabulary and key interactive tools that appear in online lessons.
Summary of Algebra II Mathematics Content

Edgenuity Algebra II strictly adheres to the content specified by the Common Core State Standards in conjunction with Louisiana Student Standards for Mathematics. Building on their work with linear, quadratic, and exponential functions, students extend their repertoire of functions to include polynomial, rational, and radical functions. Students work closely with the expressions that define the functions, and continue to expand and hone their abilities to model situations and to solve equations, including solving quadratic equations over the set of complex numbers and solving exponential equations using the properties of logarithms. The Mathematical Practice Standards apply throughout each course and, together with the content standards, prescribe that students experience mathematics as a coherent, useful, and logical subject that makes use of their ability to make sense of problem situations. The descriptions below, from the official Common Core Traditional Pathway for High School Algebra II\(^1\), summarize the areas of instruction for this course.

**Critical Area 1**

In middle school and in Algebra I, students have included integers in the study of the rational number system. In Algebra II, they move on to develop the structural similarities between the system of polynomials and the system of integers. Students draw on analogies between polynomial arithmetic and base-ten computation, focusing on properties of operations, particularly the distributive property. Students connect multiplication of polynomials with multiplication of multi-digit integers, and division of polynomials with long division of integers. Students identify zeros of polynomials, including complex zeros of quadratic polynomials, and make connections between zeros of polynomials and solutions of polynomial equations. They use the fundamental theorem of algebra to determine the number of zeros of a polynomial. A central theme of this critical area is that the arithmetic of rational expressions is governed by the same rules as the arithmetic of rational numbers.

**Critical Area 2**

Building on their previous work with functions, and on their work with trigonometric ratios and circles in geometry, students now use the coordinate plane to extend trigonometry to model periodic phenomena.

**Critical Area 3**

Students build on their knowledge about a variety of function families to synthesize and generalize what they have learned. They extend their work with exponential functions to include solving exponential equations with logarithms. They explore the effects of transformations on graphs of diverse functions, including functions arising in an application, to abstract the general principle that transformations on a graph always have the same effect regardless of the type of underlying function. They identify appropriate types of functions to model a situation, they adjust parameters to improve the model, and they compare models by analyzing appropriateness of fit and making judgments about the domain over which a model is a good fit. The description of modeling as “the process of choosing and using mathematics and statistics to analyze empirical situations, to understand them better, and to make

---

decisions” is at the heart of this critical area. The narrative discussion and diagram of the modeling cycle should be considered when knowledge of functions, statistics, and geometry is applied in a modeling context.

**CRITICAL AREA 4**

Students build on their prior experiences with data representations and statistics to relate visual displays and summary statistics to different types of data and to probability distributions. They identify different ways of collecting data—including sample surveys, experiments, and simulations—and the role that randomness and careful design play in the conclusions that can be drawn.

**STANDARDS FOR MATHEMATICAL PRACTICE IN EDGENUITY ALGEBRA II**

The Standards for Mathematical Practice complement the content standards so students increasingly engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle, and high school years. These standards are the same at all grades from kindergarten to 12th grade, but the ways in which students practice them are unique to each course.

1. **MAKE SENSE OF PROBLEMS AND PERSEVERE IN SOLVING THEM**

Students in Edgenuity Algebra II are presented with multistep and novel problems that task them with being able to make sense of problems and persevere in solving them. They employ a problem-solving process, which consists of analyzing the question, identifying and interpreting the clues within the problem, developing a strategy to solve, and checking their work. The teacher repeatedly presents this problem-solving process across Algebra II topics, especially in modeling problems—from functions to sequences. Students may check their reasoning by asking themselves “Is there another way to solve this problem?” or “Does this make sense?” At this level, students are encouraged to check their answers using a different approach.

2. **REASON ABSTRACTLY AND QUANTITATIVELY**

Abstract and quantitative reasoning are key focuses in Edgenuity Algebra II. Students continue their quantitative work from Algebra I in creating linear, quadratic, and exponential models, and extend their practice to include rational, periodic, and logarithmic functions. Students also contextualize by writing arithmetic and geometric sequences to model real-world situations. Decontextualization is emphasized in the course when students use structure to solve equations related to these models, graph functions, and translate between multiple representations of equations and expressions.

3. **CONSTRUCT VIABLE ARGUMENTS AND CRITIQUE THE REASONING OF OTHERS**

Students are tasked with explaining their reasoning and analyzing the reasoning of others throughout Edgenuity Algebra II. At this level, students explain why a concept works, challenge or defend another student’s work, explain their solution path, compare two solution paths, and justify conclusions. Students deepen their sophistication of this practice when they use important algebraic theorems to
reason about specific functions. For example, students learn and use the Fundamental Theorem of Algebra to find, describe the number of, and describe the nature of the roots of polynomial functions.

4. **Model with Mathematics**

In Edgenuity Algebra II, students extensively model real-world scenarios with equations arising from linear, quadratic, and exponential functions as in Algebra I. Now, they include modeling with logarithmic, rational, and periodic functions. They do so symbolically, graphically, and tabularly. They primarily focus on equations, but also model with inequalities, sequences, and systems. As students move through a modeling problem, they create the model, solve problems using the model, and interpret their answers in the context of the quantities they are modeling. They may ask themselves “How can this model be improved?” and place restrictions on the model based on the context of the problem.

5. **Use Appropriate Tools Strategically**

Students consider available tools when solving a mathematical problem and decide when certain tools might be helpful. In Edgenuity Algebra II, students have these options:

- Graphing calculators to manage and represent data in different forms, solve a system, evaluate logarithmic and periodic expressions, or help graph an advanced function
- Regression calculators to determine an appropriate function model
- Interactive graphs to help them reason about parameters of functions or identify the effect(s) of function transformations
- Tables to organize data and information for difficult problems
- Estimation when an exact model is not available or needed

6. **Attend to Precision**

Edgenuity Algebra II students continue to practice this standard by communicating their ideas effectively using clear and accurate mathematical language and translating between algebraic and verbal representations. They learn the meaning and purpose of new mathematical notation such as inverse function notation, and choose sensible quantities to model, solve, and interpret problems. Students are precise when they express answers to modeling problems in a manner appropriate to the context of the problem. Students concentrate on completing each step of a problem correctly, attending to units throughout the solution process.

7. **Look for and Make Use of Structure**

Students look for and make use of structure in Edgenuity Algebra II. Much like Algebra I, they use the structure of several kinds of expressions to identify ways to rewrite them. The types of expressions they work with in Algebra II become more challenging to include: recognizing and factoring quadratic in form polynomials, operations with rational expressions, solving exponential equations by rewriting the base, and translating between logarithmic and exponential forms.
8. Look for and express regularity in repeated reasoning

In Edgenuity Algebra II, students practice looking for and expressing regularity in repeated reasoning when they write arithmetic and geometric sequences both recursively and with an explicit formula. They look for repeated reasoning when they use these sequences to model real-world scenarios. They also use their investigations with these patterns to translate between recursive and explicit formulas.

Another novel way in which students practice this standard is when students learn and explore the cyclic nature of powers of the imaginary number \( i \). Students also use their extensive work in factoring polynomials to help them quickly identify ways to factor components of rational expressions to perform operations with them.

Focus in Edgenuity Algebra II

Unit 1: Linear, Quadratic, and Absolute Value Functions

Estimated Unit Time: approx. 8 Class Periods

The first unit of Algebra II builds on several functions that students worked on in Algebra I. Students determine if a function is linear and represent those that are. Linear relationships are represented in multiple forms—numerically, algebraically, and graphically. Students then move on to linear inequalities, creating one-variable inequalities and solving them (MP4). This includes solving compound inequalities and representing the solution set both algebraically and graphically. From there, students review quadratic functions and their properties. Seeing structure in a quadratic equation allows students to factor them. This leads students to finding the real solutions of a quadratic equation. Students use key attributes of a quadratic function to solve word problems. The last function that students study in this unit is the absolute value function. They analyze the graph of absolute value functions and use them to model and solve real-world problems. The final lesson of the unit covers absolute value inequalities. Students learn how to write these inequalities as compound inequalities and solve them both algebraically and graphically.

Unit 1 Focus Standards

The following focus standards are intended to guide teachers to be purposeful and strategic in both what to include and what to exclude when teaching this unit. Although each unit emphasizes certain standards, students are exposed to a number of key ideas in each unit, and as with every rich classroom learning experience, these standards are revisited throughout the course to ensure that students master the concepts with an ever-increasing level of rigor.

In the following table, green highlights indicate major work of the grade, blue highlights indicate supporting work, and yellow highlights indicate additional work.
<table>
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<th>Standard Text</th>
<th>Standard ID</th>
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<tr>
<td>Given a graph, a description of a relationship, or two input-output pairs (include reading these from a table), construct linear and exponential functions, including arithmetic and geometric sequences, to solve multistep problems.</td>
<td>A2:F-LE.A.2</td>
</tr>
<tr>
<td>Define appropriate quantities for the purpose of descriptive modeling.</td>
<td>A2:N-Q.A.2</td>
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<tr>
<td>Interpret the parameters in a linear, quadratic, or exponential function in terms of a context.</td>
<td>A2:F-LE.B.5</td>
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<tr>
<td>Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions.</td>
<td>A2:A-CED.A.1</td>
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<tr>
<td>Use the structure of an expression to identify ways to rewrite it. <strong>For example, see</strong> $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.</td>
<td>A2:A-SSE.A.2</td>
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<tr>
<td>Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers $a$ and $b$.</td>
<td>A2:A-REI.B.4.b</td>
</tr>
<tr>
<td>Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.</td>
<td>A2:F-IF.C.7.b</td>
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<tr>
<td>Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k f(x)$, $f(kx)$, and $f(x + k)$ for specific values of $k$ (both positive and negative); find the value of $k$ given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.</td>
<td>A2:F-BF.B.3</td>
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<tr>
<td>Explain why the $x$-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.</td>
<td>A2:A-REI.D.11</td>
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## Unit 1 Pacing Guide

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<td>• Describe if a function is linear.</td>
<td>A2:F-LE.A.2</td>
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<td>• Represent a linear relationship numerically, algebraically, and graphically.</td>
<td>A2:N-Q.A.2</td>
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<td>A2:F-LE.B.5</td>
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<tr>
<td>Inequalities</td>
<td>• Solve one-variable linear inequalities, including compound inequalities,</td>
<td>A2:A-CED.A.1</td>
<td>1</td>
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<td>and represent the solution sets graphically and algebraically.</td>
<td>A2:N-Q.A.2</td>
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<td></td>
<td>• Create one-variable linear inequalities in one variable and use them to</td>
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<td>solve problems.</td>
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<tr>
<td>Quadratic Functions</td>
<td>• Find the line of symmetry and vertex of a parabola given its function</td>
<td>A2:F-LE.B.5</td>
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<td>rule.</td>
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<td>• Identify a quadratic function from the function rule.</td>
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<td>• Use key attributes of a quadratic function to solve word problems.</td>
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<tr>
<td>Solving Quadratic</td>
<td>• Find real solutions for quadratic equations using the zero product</td>
<td>A2:A-SSE.A.2</td>
<td>1.5</td>
</tr>
<tr>
<td>Equations by Factoring</td>
<td>property.</td>
<td>A2:A-REI.B.4.b</td>
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<tr>
<td></td>
<td>• Use key attributes of a quadratic function to solve word problems.</td>
<td>A2:F-LE.B.5</td>
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<tr>
<td>Absolute Value Functions</td>
<td>• Analyze absolute value functions to determine key features of the graph.</td>
<td>A2:F-IF.C.7.b</td>
<td>1</td>
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<td>• Model and solve mathematical and real-world problems with absolute value</td>
<td>A2:F-BF.B.3</td>
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<td></td>
<td>functions.</td>
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<tr>
<td>Absolute Value</td>
<td>• Rewrite absolute value inequalities as compound inequalities.</td>
<td>A2:A-REI.D.11</td>
<td>1.5</td>
</tr>
<tr>
<td>Inequalities</td>
<td>• Solve absolute value inequalities graphically and algebraically.</td>
<td>A2:F-IF.C.7.b</td>
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### Discussion Questions & Answers

1. What are the steps for solving a quadratic equation by factoring?

   a. **The first step in solving a quadratic equation by factoring** is to write the equation in standard form, so that one side of the equation is equal to zero. Next, factor the expression. Then you can use the zero product property and set each factor equal to zero. Solve each equation and check the solutions.

2. Describe how to use a graphing calculator to solve the inequality $|x - 6| - 1 \geq 0$.

   a. **First, you need to isolate the absolute value expression, so add 1 to both sides.** That leaves $|x - 6| \geq 1$. Next, **graph each side of the inequality as a function**. So we will graph $y = |x - 6|$.
and \( y = 1 \) in the graphing calculator. Then we identify the region that satisfies the inequality. In this case, we need the region of the absolute value function that is greater than or equal to the horizontal line. The end points where this occurs is where \( x = 5 \) and \( x = 7 \). Since the inequality is greater than and equal to, we include the endpoints. The solution is \( x \leq 5 \) and \( x \geq 7 \).

3. The inequality \(|x + 4| < -2\) does not have any solutions. Explain why, using both algebraic and graphical arguments.
   a. An absolute value expression represents distance from 0. Since distance is always a positive value, it cannot be less than a negative value, like \(-2\).
   b. Graphically, if you graph the system of equations \( y = |x + 4| \) and \( y = -2 \), you will see that \( y = |x + 4| \) is a “v”-shaped graph centered at \((-4, 0)\) and \( y = -2 \) is a horizontal line underneath it. The graphs do not intersect, which indicates that there is no solution.

Common Misconceptions

- **Slope of a line**
  - Students calculate slope as change in \( x \) over change in \( y \).
  - Students mismatch coordinates.
    - E.g., if students are calculating the slope between points \((1, 2)\) and \((3, 4)\) they write \( m = \frac{4-2}{1-3} \)
  - Students see slope as a special letter “m,” or memorized “rise over run” and not quantification of the change in the dependent variable per change in the independent variable.

- **Slope-intercept form of a line**
  - Students will take the first number listed as slope and the last number as the \( y \)-intercept.
  - Students will not recognize that when linear equations are not in slope-intercept form the coefficient on \( x \) is not the slope.

- **Inequalities**
  - Students do not use the opposite inequality when multiplying or dividing by a negative number.
  - Students use the opposite inequality when subtracting or adding a negative number to both sides of an inequality.
  - Students believe that the inequality symbol is an “arrow” for the solution graphed on the number line.
  - Students incorrectly state solutions of inequalities as a single number instead of a set of numbers.

- **Quadratic functions**
  - Students believe the axis of symmetry for a parabola is just a number and not the equation for a line.
  - Students factor to solve and neglect to apply the zero product property.
    - E.g. \( 0 = (x - 2)(x + 1) \), so the solutions are incorrectly given as \( x = -2 \) and \( x = 1 \).
• Students use a formula to find the x-value of the vertex, but do not understand they must substitute that value into the quadratic equation to find the y-value of the vertex.
• Students do not always internalize the meaning of an intercept.
• Students do not check their answers when solving a real-world problem in terms of the context.
• Students do not recognize a quadratic equation if it is not in $ax^2 + bx + c$ form.

- **Absolute value functions and inequalities**
  - Students shift functions incorrectly when a constant is added or subtracted from the input value, i.e., $f(x + c)$ shifts left and not right or $f(x – c)$ shifts right and not left.
  - Students make errors when identifying the vertex of an absolute value function given in the form $f(x) = a|x – h| + k$
    e.g., they identify the vertex as $(k, h)$ instead of $(h, k)$.
  - Students do not create two equations or inequalities when solving an absolute value equation or inequality.
  - Students do not isolate the absolute value expression before creating two inequalities.

**Classroom Challenge**

**Part I:**
A quadratic of the form $x^2 + bx + c = 0$ is factored into two binomials $(x + k)$ and $(x + m)$.

  i. Write an expression for the sum of the zeros of the function in terms of $b$ and/or $c$.
  ii. Write an expression for the product of the zeros of the function in terms of $b$ and/or $c$.

**Part II:**
A quadratic of the form $ax^2 + bx + c = 0$, where $a$ is a value other than 0 or 1, is factored into two binomials $(rx + k)$ and $(qx + m)$.

  i. How does the expression for the sum of the zeros of the function change when $a$ is included in the quadratic?
  ii. How does the expression for the product change?

**Possible solution pathway:**

**Part I:**
When the two binomials are multiplied, the result is $x^2 + kx + mx + km$. This can be written as $x^2 + x(k + m) + km$.

  i. The sum of the values $k$ and $m$ is equal to the value of $b$. 
In the equation \((x + k)(x + m) = 0\), \(x = -k\) or \(-m\), the zeros are actually the opposites of \(k\) and \(m\). Therefore, the sum of the zeros of the function is equal to \(-b\).

ii. The zeros of the function are \(x = -k\) or \(-m\), so the product is \(-k \cdot -m\) or \(km\). In the result of the multiplication shown above, \(km\) is the constant in the quadratic, or the \(c\) value.

Part II:
The equation \(ax^2 + bx + c = 0\) can be rewritten as \(a(x^2 + \frac{b}{a}x + \frac{c}{a}) = 0\) by factoring. This means that:
- the sum of the zeros is now \(-\frac{b}{a}\) instead of just \(-b\), and
- the product of the zeros is now \(-\frac{c}{a}\) instead of just \(c\).

**Additional possible solution pathway:**

Part I:
Let’s substitute values in for \(b\) and \(c\). Let \(b = 5\) and \(c = 6\).

\[x^2 + 5x + 6 = 0\]

In this case, the factored form of the equation is \((x + 3)(x + 2) = 0\), which means the zeros are \(-2\) and \(-3\). The sum of \(-2\) and \(-3\) is \(-5\), which is the opposite of the \(b\) value or \(-b\). The product of \(-2\) and \(-3\) is \(6\), which is the \(c\) value.

Part II:
This time, let’s substitute values for \(a\), \(b\), and \(c\), being careful to choose numbers that will result in a factorable equation. Let \(a = 15\), \(b = 11\), and \(c = 2\).

\[15x^2 + 11x + 2 = 0\]

In this case, the factored form of the equation is \((3x + 1)(5x + 2) = 0\), which means the zeros are \(-\frac{1}{3}\) and \(-\frac{2}{5}\).

The sum of \(-\frac{1}{3}\) and \(-\frac{2}{5}\) is \(-\frac{11}{15}\). This is the opposite of the \(b\) value, like in the Part 1, but it is also divided by the \(a\) value. The sum of the zeros is \(-\frac{b}{a}\).
The product of $-\frac{1}{3}$ and $-\frac{2}{5}$ is $\frac{2}{15}$. This is the $c$ value, like in Part I, but divided by the $a$ value. The product of the zeros is $\frac{c}{a}$.

Teacher Notes:

Students who have a hard time getting started may benefit from being asked to take the factored form of the quadratic and multiply the two binomials together. This will allow the students, especially in Part I, to see how the values in the factored form related directly to the values in the standard form of the equation.

Students may benefit from verbally telling where the values are derived from while receiving prompting as to how this relates to the variable expression for each portion. Additionally, students may benefit from experimenting with numerical values as shown in the additional solution pathway.

Watch for students who use the values of $k$ and $m$ as the zeros in Part I and for students who use $m/q$ or $k/r$ as the zeros in Part II.

More advanced students may benefit from applications of the products and sums of zeros properties by asking them to find the zeros of factorable quadratic equations with many possible factor pairs for both $a$ and $c$. These equations may contain variable expressions for any coefficient or constant.

UNIT 2: RELATIONSHIPS BETWEEN QUANTITIES

Estimated Unit Time: approx. 14 Class Periods

This unit begins with students using problem-solving strategies to create and solve equations for a variety of word problems, including mixture, time-distance-rate, and work problems (MP1, MP2). Students move on to solving literal equations in terms of a given variable, using properties of equality. Next, they look at functions and represent them in multiple ways, including creating models for real-world scenarios (MP4). Students combine functions using arithmetic operations and evaluate these sums, differences, products, and quotients. Composition of functions is introduced and students write expressions for the compositions. Students learn how to find the inverse of a function. Then they study the rate of change of a function by calculating and interpreting rate of change. A performance task ties it all together to conclude the unit, incorporating several practice standards (MP1, MP2, MP3, and MP6) within the content standards covered in this project.

Unit 2 Focus Standards

The following focus standards are intended to guide teachers to be purposeful and strategic in both what to include and what to exclude when teaching this unit. Although each unit emphasizes certain
standards, students are exposed to a number of key ideas in each unit, and as with every rich classroom learning experience, these standards are revisited throughout the course to ensure that students master the concepts with an ever-increasing level of rigor.

In the following table, green highlights indicate major work of the grade, blue highlights indicate supporting work, and yellow highlights indicate additional work.

<table>
<thead>
<tr>
<th>Standard Text</th>
<th>Standard ID</th>
</tr>
</thead>
<tbody>
<tr>
<td>Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.</td>
<td>A2:A-CED.A.1</td>
</tr>
<tr>
<td>Determine an explicit expression, a recursive process, or steps for calculation from a context.</td>
<td>A2:N-Q.A.2</td>
</tr>
<tr>
<td>Explain each step in solving an equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.</td>
<td>A2:A-REI.A.1</td>
</tr>
<tr>
<td>Determine an explicit expression, a recursive process, or steps for calculation from a context.</td>
<td>A2:F-BF.A.1.a</td>
</tr>
<tr>
<td>Combine standard function types using arithmetic operations. For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.</td>
<td>A2:F-BF.A.1.b</td>
</tr>
<tr>
<td>Solve an equation of the form ( f(x) = c ) for a simple function ( f ) that has an inverse and write an expression for the inverse. For example, ( f(x) = 2x^3 ) or ( f(x) = \frac{x+1}{(x-1)} ) for ( x \neq 1 ).</td>
<td>A2:F-BF.B.4.a</td>
</tr>
<tr>
<td>For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.</td>
<td>A2:F-IF.B.4</td>
</tr>
<tr>
<td>Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.</td>
<td>A2:F-IF.B.6</td>
</tr>
<tr>
<td>Given a graph, a description of a relationship, or two input-output pairs (include reading these from a table), construct linear and exponential functions, including arithmetic and geometric sequences, to solve multi-step problems.</td>
<td>A2:F-LE.A.2</td>
</tr>
<tr>
<td>Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions.</td>
<td>A2:A-CED.A.1</td>
</tr>
</tbody>
</table>
## Unit 2 Pacing Guide

<table>
<thead>
<tr>
<th>Lesson</th>
<th>Objectives</th>
<th>Standards</th>
<th>Number of Days</th>
</tr>
</thead>
</table>
| Word Problems               | • Create equations to solve a variety of word problems such as mixture, time-distance-rate, and work.  
• Solve a variety of word problems, and interpret the solutions in context. | A2:A-CED.A.1      | 1              |
|                             |                                                                             | A2:N-Q.A.2        |                |
| Literal Equations           | • Solve a literal equation in terms of a given variable.  
• Determine if expressions are equivalent.                                             | A2:A-REI.A.1      | 1.5            |
| Relations and Functions     | • Represent a relation in multiple ways, including equations, graphs, words, and tables of values.  
• Determine if a relation is a function.  
• Determine if the function is one-to-one.  
• Determine the domain and range of a relation.  
• Evaluate function rules.                                                               | A2:F-BF.A.1.a    | 2              |
|                             |                                                                             | A2:N-Q.A.2        |                |
| Function Operations         | • Combine functions using arithmetic operations, expressing the results both algebraically and graphically.  
• Evaluate sums, differences, products, and quotients of functions.                   | A2:F-BF.A.1.b     | 1              |
| Composition of Functions    | • Write an expression for the composition of functions.  
• Find the domain of the composition of functions.  
• Evaluate the composition of functions.                                                | A2:F-BF.A.1.a     | 1.5            |
| Function Inverses           | • Find the inverse of a function.  
• Use composition to verify that functions are inverses.                               | A1:F-BF.B.3       | 1.5            |
| Rate of Change              | • Calculate the average rate of change of a function over a specified interval.  
• Interpret the average rate of change of a function over a specified interval.  
• Solve problems involving direct variation.                                           | A2:F-IF.B.4      | 1.5            |
|                             |                                                                             | A2:F-IF.B.6       |                |
| Performance Task: Going on a Round Trip |                                                                             | A2:F-BF.A.1.a     | 3              |
|                             |                                                                             | A2:A-REI.A.1      |                |
|                             |                                                                             | A2:F-LE.A.2       |                |
|                             |                                                                             | A2:A-CED.A.1      |                |
Discussion Questions & Answers

1. Explain each step in solving the equation \( \frac{4xy}{5} - 2z = \frac{xy}{3} \) for the variable \( x \).
   - First, multiply all terms by 15, a common multiple of 3 and 5, to clear the fractions. This leaves the equation \( 3(4xy) - 15(2z) = 5(xy) \). We can simplify that by multiplying to get \( 12xy - 30z = 5xy \). Now let’s get the terms that contain an \( x \) on one side of the equation and all other terms on the other side of the equation. We can do that by adding 30z and subtracting 5xy from both sides. That leaves the equation \( 12xy - 5xy = 30z \). Factoring out an \( x \) on the left side of the equation gives us \( x(12y - 5y) = 30z \). The terms in the parenthesis are like terms, so they can be combined, leaving us with \( x(7y) = 30z \). Finally divide both sides by 7y to isolate \( x \). The solution is \( x = \frac{30z}{7y} \).

2. How are standard function types combined using arithmetic operations (addition, subtraction, multiplication, division)? Discuss how to do so from a rule, a table, and a graph.
   - To combine functions given a rule, you add, subtract, multiply, or divide the algebraic expressions. It is important to pay close attention to distributing when subtracting and multiplying functions.
   - To combine functions from a table, you add, subtract, multiply, or divide the \( y \)-values having the same \( x \)-value.
   - To combine functions from a graph, you add, subtract, multiply, or divide the \( y \)-coordinates having the same \( x \)-coordinate.

3. Consider a function that relates the speed of a car, in miles per hour, to time, in minutes. If the average rate of change from 2 to 6 is a positive value, what does that mean in terms of the car’s speed over this time interval? What if the average rate of change were 0? What if the average rate of change were a negative value?
   - If the average rate of change over the interval from 2 to 6 is positive, it means that the car was traveling at a faster speed at 6 minutes, then at 2 minutes.
   - If the average rate of change over the interval from 2 to 6 is 0, it means that the car was traveling at the same speed at 2 minutes and at 6 minutes.
   - If the average rate of change over the interval from 2 to 6 is negative, it means that the car was traveling at a faster speed at 2 minutes than at 6 minutes.

Common Misconceptions

- Solving Equations
  - Students might neglect the negative solution when solving equations using the square root property.
  - Students incorrectly apply the distributive property, not attending to the fact that it is multiplication over addition or subtraction.
  - Students forget to change all the signs of the terms when subtracting a quantity or distributing/factoring a negative number.
Students may neglect the order of operations, especially when subtraction precedes a term distribution.

- Relations and Functions
  - Students consider \( f(x) \) as an operation instead of a function, \( f \), defined in terms of \( x \).
  - Students might switch domain and range for a relation or function.
  - Students may think that each value in the range of a function must map back to a single value in the domain.
  - Students might not connect \( f(0) \) with the \( y \)-intercept of a function.

- Combining Functions
  - Students might try to combine terms that are not like or change the degree of variables when combining terms.
  - Students do not attend to order when finding the composition of functions. E.g., they find \( g \circ f \) when asked to find \( f \circ g \).
  - Students may confuse function composition with function multiplication.

- Function Inverses
  - Students might just distribute a negative 1 to the equation of a function to find its inverse.
  - Students might believe all functions have inverses and need to see counterexamples.
  - Students may look for reflections of the graphs of functions to identify inverse, but they do not realize that it must be a reflection about the line \( x = y \).
  - Students might assume if a relationship is a function, then its inverse is also a function.

- Rate of change
  - Students calculate the rate of change as change in \( x \) over change in \( y \).
  - Students do not recognize that the rate of change for a line is its slope.
  - Students may think that one cannot calculate the average rate of change for a nonlinear function.
  - Students think average rate of change has to always be positive.
  - Students cannot interpret rate of change or average rate of change for a real-world context.

Classroom Challenge
Sam is shopping for new shoes. He has a coupon for $5 off a pair of shoes, and when he gets to the store he also sees that all shoes will be discounted 20% off. Let \( x \) represent the original price of a pair of shoes.

Part I:
Create a function \( f(x) \) that represents the price of the shoes after the coupon is applied.
Create a function \( g(x) \) that represents the price of the shoes after the 20% discount is applied.
Part II:
Write two more functions \( p(x) \) and \( q(x) \) in terms of the functions \( f(x) \) and/or \( g(x) \), where
- \( p(x) \) represents the price of the shoes if the coupon is applied first and then the discount is applied.
- \( q(x) \) represents the price of the shoes if the discount is applied first and then the coupon is applied.

Part III:
Sam finds a pair of shoes that costs $42. The store asks him which order he wants the coupon and discount applied. Which order should he pick? Why?

**Possible solution pathway:**
Part I:
\[
\begin{align*}
  f(x) &= x - 5 \\
  g(x) &= x - 0.2x = 0.8x
\end{align*}
\]
Part II:
\[
\begin{align*}
  p(x) &= 0.8f(x) \\
  q(x) &= g(x) - 5
\end{align*}
\]
Part III:
\[
\begin{align*}
  p(42) &= 0.8f(42) = 0.8(42 - 5) = 29.6 \\
  q(42) &= g(42) - 5 = 0.8(42) - 5 = 28.6
\end{align*}
\]
Sam should have the store apply the 20% discount first and then the $5 coupon. If this order is applied, the shoes will cost him $28.60. If the other order is applied, it would cost him $29.60—a dollar more!

**Teacher Notes:**
Students who struggle with Part II should write the function without considering \( f(x) \) and \( g(x) \) first and then try to use function operations to create the functions. They might benefit from choosing a real value and thinking about how the discounts affect that value.

For advanced students, ask them to explain why the order matters and why evaluating \( p(x) \) and \( q(x) \) at the same value will result in different prices. Ask them to graph the functions and interpret the graphs.

**UNIT 3: QUADRATICS AND COMPLEX NUMBERS**

*Estimated Unit Time: approx. 12 Class Periods*

In this unit, students extend their knowledge of quadratic functions by exploring imaginary roots. Students investigate properties of complex numbers (MP8) and use properties to perform operations with complex numbers. Methods of factoring, completing the square, and the quadratic formula are used to solve quadratic equations over the complex number system. Students use quadratic equations to model and solve real-world problems (MP4). Studying transformations of the quadratic function,
students must use the structure of quadratics to identify and graph transformations of the parent function. They finish this unit by rewriting square root and cube root expressions to graph and transform the graphs of the parent functions.

**Unit 3 Focus Standards**

The following focus standards are intended to guide teachers to be purposeful and strategic in both what to include and what to exclude when teaching this unit. Although each unit emphasizes certain standards, students are exposed to a number of key ideas in each unit, and as with every rich classroom learning experience, these standards are revisited throughout the course to ensure that students master the concepts with an ever-increasing level of rigor.

In the following table, green highlights indicate major work of the grade, blue highlights indicate supporting work, and yellow highlights indicate additional work.

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<tbody>
<tr>
<td>Know there is a complex number ( i ) such that ( i^2 = -1 ), and every complex number has the form ( a + bi ) with ( a ) and ( b ) real.</td>
<td>A2:N-CN.A.1</td>
</tr>
<tr>
<td>Use the relation ( i^2 = -1 ) and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.</td>
<td>A2:N-CN.A.2</td>
</tr>
<tr>
<td>Use the structure of an expression to identify ways to rewrite it. For example, see ( x^4 - y^4 ) as ((x^2)^2 - (y^2)^2), thus recognizing it as a difference of squares that can be factored as ((x^2 - y^2)(x^2 + y^2)).</td>
<td>A2:A-SSE.A.2</td>
</tr>
<tr>
<td>Solve quadratic equations by inspection (e.g., for ( x^2 = 49 )), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as ( a \pm bi ) for real numbers ( a ) and ( b ).</td>
<td>A2:REI.B.4.b</td>
</tr>
<tr>
<td>Solve quadratic equations with real coefficients that have complex solutions.</td>
<td>A2:N-CN.C.7</td>
</tr>
<tr>
<td>Determine an explicit expression, a recursive process, or steps for calculation from a context.</td>
<td>A2:F-BF.A.1.a</td>
</tr>
<tr>
<td>Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions.</td>
<td>A2:A-CED.A.1</td>
</tr>
<tr>
<td>Identify the effect on the graph of replacing ( f(x) ) by ( f(x) + k ), ( k f(x) ), ( f(kx) ), and ( f(x + k) ) for specific values of ( k ) (both positive and negative); find the value of ( k ) given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.</td>
<td>A2:F-BF.B.3</td>
</tr>
</tbody>
</table>
Standard Text | Standard ID
---|---
Rewrite expressions involving radicals and rational exponents using the properties of exponents. | A2:N-RN.A.2
Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions. | A2:F-IF.C.7.b

**Unit 3 Pacing Guide**

<table>
<thead>
<tr>
<th>Lesson</th>
<th>Objectives</th>
<th>Standards</th>
<th>Number of Days</th>
</tr>
</thead>
</table>
| Complex Numbers | • Represent square roots of negative numbers as multiples of \(i\).  
• Represent complex numbers in the form \(a + bi\) or in the complex plane.  
• Simplify powers of \(i\) using their cyclic nature.  
• Determine the absolute value of a complex number. | A2:N-CN.A.1 | 1 |
| Performing Operations with Complex Numbers | • Perform addition, subtraction, multiplication, and division of complex numbers.  
• Identify the field properties of complex numbers. | A2:N-CN.A.2 | 1.5 |
| Completing the Square | • Recognize the pattern of a perfect-square trinomial as the square of a binomial.  
• Use the square root property to solve equations.  
• Find complex solutions to quadratic equations by completing the square. | A2:A-SSE.A.2  
A2:A-REI.B.4.b  
A2:N-CN.C.7 | 2 |
| The Quadratic Formula | • Find real and complex solutions of quadratic equations using the quadratic formula.  
• Use the discriminant to determine the number and type of roots of a quadratic equation. | A2:A-REI.B.4.b  
A2:N-CN.C.7 | 1 |
| Modeling with Quadratic Equations | • Use quadratic equations to model and solve real-world problems. | A2:F-BF.A.1.a  
A2:A-REI.B.4.b  
A2:A-CED.A.1 | 1 |
| Transformations of Quadratic Functions | • Use completing the square to write quadratic functions in the form \(y = a(x - h)^2 + k\).  
• Describe the effects of changes in \(a\), \(h\), and \(k\) to the graph of a function in the form \(y = a(x - h)^2 + k\). | A2:A-SSE.A.2  
A2:F-BF.B.3 | 1 |
Lesson Objectives Standards Number of Days

The Square Root Function
- Simplify a square root whose radicand is a perfect square.
- Graph the square root function and reflections over the axes.
- State the domain and range of square root functions.
A2:N-RN.A.2 A2:F-IF.C.7.b 2

The Cube Root Function
- Graph the cube root function, and translations and reflections of it.
- State the key features of the cube root function, and translations and reflections of it.

Unit Test

Discussion Questions & Answers

1. Describe the similarities and differences between the graphs of \( y = x^2 \) and \( y = (x - a)^2 + b \). Assume \( a \) and \( b \) are positive. What could you change about one of the graphs so that they have nothing in common?
   a. Both graphs will be parabolas that open upward. The graph of \( y = x^2 \) will have a vertex at the origin and an axis of symmetry of \( x = 0 \). The graph of \( y = (x - a)^2 + b \) will be shifted 2 units to the right and 3 units up from the parent graph. That means its vertex will be at \((2, 3)\) and the axis of symmetry is \( x = 2 \). I could add a negative 1 in front of one of the equations so that one parabola opens downward.

2. Explain the process of rewriting \( x^2 - 8x + 23 = 0 \) as \((x - 4)^2 = -7\). Why is it helpful to rewrite it this way?
   a. To rewrite the equation, we can complete the square. First we can subtract 23 from both sides of the equation to get \( x^2 - 8x = -23 \). Then we need to determine the value to add to both sides that will create a perfect square trinomial on the left side of the equation. We do this by taking half of the \( b \)-term and squaring it to get \((-4)^2 \) or 16. Now we add 16 to both sides of the equation to get \( x^2 - 8x + 16 = -23 + 16 \). We factor the left side and simplify the right side to get \((x - 4)^2 = -7\). It is helpful to rewrite a quadratic equation this way because it helps us solve the quadratic equation using the square root property.

3. How do you simplify square root expressions that do not have a perfect square in the radicand? Can you use this method for cube root expressions?
   a. First, write the prime factorization of the radicand. Then you can apply the product property. Write the radicand as a product, forming as many perfect squares as possible. Then simplify each perfect square and find the product. You can use this method for cube root expressions except you have to look for perfect cubes instead of perfect squares.
Common Misconceptions

- **Complex Numbers**
  - Students might confuse complex numbers with irrational numbers.
  - Students may think that $i$ is a variable.
  - Students incorrectly combine real and imaginary parts of complex numbers.
  - Students forget to distribute the negative to both parts of a complex number when subtracting.
  - Students might write the conjugate of $a + bi$ as $-a - bi$.
  - Students may not realize that $i^2$ can be further simplified to $-1$.

- **Solving Quadratics**
  - Students might neglect the negative solution when solving equations using the square root property.
  - Students may not isolate the expression being squared before attempting to take the square root.
  - Students omit 0 as a solution.
    - E.g., if they are solving $x(x - 1) = 0$, students might divide out the $x$ factor or just focus on the $(x - 1)$ factor.
  - Students forget to add the value that completes the square to both sides of the equation.
  - Students might forget or confuse parts of the quadratic formula.
  - Students may substitute incorrect values into the quadratic formula if the quadratic equation is not in standard form.
  - Students neglect the real-world context when giving solutions for a quadratic model.

- **Graphing Quadratic, Square Root, and Cube Root Functions**
  - Students shift functions incorrectly when a constant is added or subtracted from the input value, i.e., $f(x + c)$ shifts left and not right or $f(x - c)$ shifts right and not left.
  - Students take half of the radicand when simplifying a square root expression.
  - Students may think any number times 2 is a perfect square and any number times 3 is a perfect cube.
Classroom Challenge
The diagram below shows a geometric representation of completing the square. The expressions inside each shape represent the area of the shape.

Diagram A:

Diagram B:

Part I: In terms of $b$ and $x$, write an expression for the area of the composite figure in Diagram A. Write your answer as a simplified binomial.
Part II: In terms of \( b \) and \( x \), write an expression for the area of the composite figure in Diagram B. Write your answer as the square of a binomial using the dimensions of the large square. Explain your solution.

Part III: Diagrams A and B differ only by the areas of the four squares in the corner of the figure. Using the expression from Parts I and II, write an equation showing how the expression for the areas of Diagram A and Diagram B are related.

Part IV: How can the diagram be used to solve the equation \( x^2 + 8x = 9 \) for the positive value of \( x \)?

Possible Solution Pathway:

Part I: An expression that represents the area of the figure is \( x^2 + 4\left(\frac{1}{4}bx\right) \) or \( x^2 + bx \).

Part II: An expression that represents the area of the large square depends on the dimensions of the square.

Because the area of the square in the middle is \( x^2 \), the side length of the center square is \( x \).

Because each rectangle shares a side with the small square, one side of the rectangle is \( x \). Since the area of each rectangle is \( \frac{1}{4}bx \), the other dimension of the rectangle must be \( \frac{1}{4}b \).

One side of the large square is made up of two short sides of the rectangles and one long side, or \( 2\left(\frac{1}{4}b\right) + x \). This can be written as \( x + \frac{1}{2}b \). To find the area, I need to square the side length, so \( \left(x + \frac{1}{2}b\right)^2 \) is the expression that represents the area of the large square.

Part III: The area of one of the small squares is the square of the short side of the rectangle, \( \frac{1}{4}b \), so \( \frac{1}{16}b^2 \). There are four small squares. The area of all four of them is \( \frac{1}{4}b^2 \). This is how much larger Diagram B is than Diagram A, so the equation relating the two is Diagram A = Diagram B – the area of the four small squares.

\[
x^2 + bx = \left(x + \frac{1}{2}b\right)^2 - \frac{1}{16}b^2
\]

Part IV:

If \( x^2 + 8x = 9 \), then \( b = 8 \).
Use a figure similar to Figure A to show $x^2 + 8x$. The area of the large square would be $x^2$, while the areas of the four rectangles would be $8x$, making the area of one rectangle $2x$.

This means the non-$x$ side length of the rectangle is $2$.

Each small square that would “complete the square” has a side length of $2$, so an area of $4$ square units. Since there are four small squares, that’s a total area of $16$ square units being subtracted from the large square.

This means that the area of the large square, $(x + 4)^2$ decreased by $16$ is equal to the original area $x^2 + 8x$ or $9$.

\[
(x + 4)^2 - 16 = 9 \\
(x + 4)^2 = 25 \\
x + 4 = 5 \\
x = 1
\]

**Teacher Notes:**

A student who has a hard time getting started on Parts I and II may benefit from labeling the lengths on the diagrams, beginning with the inner-most square.

Students who have a hard time manipulating the variable expressions in Parts I, II, and III may benefit from a concrete example using numbers instead of variables. Students may then be able to transfer the concrete example to the more abstract.

Students who may struggle with using the diagram to determine the value of $x$ in Part IV may benefit from labeling the areas on a diagram similar to Figure A before beginning. These students may also benefit by reviewing the transitive property since one expression, $x^2 + 8x$, is both equal to $9$ and the expression that results from completing the square.
More advanced students may benefit from using the diagram to show solutions that are irrational. These students may also be challenged to explain how the second solution to the quadratic in Part IV exists in relation to the diagram.

UNIT 4: SYSTEMS OF EQUATIONS

Estimated Unit Time: approx. 9 Class Periods

In this unit, students extend their knowledge of two-variable linear inequalities and systems to include quadratic equations and 3 x 3 systems. They solve systems of linear equations and systems of linear inequalities graphically. Elimination and substitution are also used, with students looking at the structure of the equations to set up elimination properly and decide which method is most efficient given the system. Systems of equations are used to model and solve problems (MP4). Students solve linear and quadratic equations by graphing a related system of equations (MP5).

Unit 4 Focus Standards

The following focus standards are intended to guide teachers to be purposeful and strategic in both what to include and what to exclude when teaching this unit. Although each unit emphasizes certain standards, students are exposed to a number of key ideas in each unit, and as with every rich classroom learning experience, these standards are revisited throughout the course to ensure that students master the concepts with an ever-increasing level of rigor.

In the following table, green highlights indicate major work of the grade, blue highlights indicate supporting work, and yellow highlights indicate additional work.

<table>
<thead>
<tr>
<th>Standard Text</th>
<th>Standard ID</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solve systems of linear equations exactly and approximately (e.g., with graphs), limited to systems of at most three equations and three variables. With graphic solutions, systems are limited to two variables.</td>
<td>A2:A-REI.C.6</td>
</tr>
<tr>
<td>Define appropriate quantities for the purpose of descriptive modeling.</td>
<td>A2:N-Q.A.2</td>
</tr>
<tr>
<td>Solve quadratic equations by inspection (e.g., for ( x^2 = 49 ), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as ( a \pm bi ) for real numbers ( a ) and ( b ).</td>
<td>A2:A-REI.B.4.b</td>
</tr>
<tr>
<td>Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. For example, find the points of intersection between the line ( y = -3x ) and the circle ( x^2 + y^2 = 3 ).</td>
<td>A2:A-REI.C.7</td>
</tr>
</tbody>
</table>
Standard Text

Explain why the x-coordinates of the points where the graphs of the equations \( y = f(x) \) and \( y = g(x) \) intersect are the solutions of the equation \( f(x) = g(x) \); find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where \( f(x) \) and/or \( g(x) \) are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.

Standard ID

A2:A-REI.D.11

## Unit 4 Pacing Guide

<table>
<thead>
<tr>
<th>Lesson</th>
<th>Objectives</th>
<th>Standards</th>
<th>Number of Days</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solving Linear Systems Graphically</td>
<td>• Solve systems of two-variable linear equations graphically.</td>
<td>A2:A-REI.C.6</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>• Classify systems of two-variable equations as dependent, independent, consistent, or inconsistent.</td>
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<tr>
<td></td>
<td>• Solve systems of two-variable linear inequalities.</td>
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<td></td>
</tr>
<tr>
<td>Solving Linear Systems by Elimination</td>
<td>• Solve systems of two-variable linear equations using elimination.</td>
<td>A2:A-REI.C.6</td>
<td>1</td>
</tr>
<tr>
<td>Solving Linear Systems by Substitution</td>
<td>• Solve systems of two-variable linear equations using substitution.</td>
<td>A2:A-REI.C.6</td>
<td>1</td>
</tr>
<tr>
<td>Solving 3 x 3 Linear Systems</td>
<td>• Classify systems of three-variable equations as dependent, independent, consistent, or inconsistent.</td>
<td>A2:A-REI.C.6</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>• Solve 3 × 3 linear systems algebraically.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Solving Linear-Quadratic Systems</td>
<td>• Solve a system of equations consisting of a line and a parabola algebraically and graphically, using technology where appropriate.</td>
<td>A2:A-REI.B.4.b A2:A-REI.C.7</td>
<td>1</td>
</tr>
<tr>
<td>Solving One-Variable Equations with Systems</td>
<td>• Solve a one-variable linear or quadratic equation by graphing a related system of equations.</td>
<td>A2:A-REI.D.11 A2:A-REI.C.6 A2:A-REI.C.7</td>
<td>2</td>
</tr>
</tbody>
</table>
Discussion Questions & Answers

1. What is the relationship between the solution(s) to a system of equations and its graphical representation?
   a. The solution(s) to a system of equations is all the values that make both equations true. Graphically, the solutions to a system of equations are the intersection point(s) of the graphs of both equations.

2. Name the three types of systems of linear equations and describe the graphs for each.
   a. Consistent and independent systems of linear equations intersect once, so they have one solution.
   b. Consistent and dependent systems of linear equations are the same line, so they have infinite solutions.
   c. Inconsistent systems of linear equations do not intersect (they are parallel lines), so there are no solutions.

3. Compare and contrast the different methods to solve a system of linear equations. Do these same methods work for solving a linear-quadratic system?
   a. There are three different ways you can solve a system of linear equations. One way is to graph both equations of a system and determine the intersection point(s). Sometimes the graphs do not intersect at a gridline, in which case, you can only approximate the solutions. Another way to solve a system of linear equations is elimination. You add or subtract equations to eliminate one of the variables. Sometimes you need to multiply one or both equations in order for the x- or y-coefficient terms to be eliminated. You can also use substitution to solve a system of linear equations. You solve for one of the variables in one of the equations and substitute the expression into the other.
   b. You can use all of these methods to solve a linear-quadratic system as well.

Common Misconceptions

- Graphs of Systems of Equations
  o Students might not realize a point must lie on both graphs to be a solution to the system.
  o Students do not realize a system of linear equations can have one solution, no solution, or infinite solutions.
  o Students do not realize a linear-quadratic system of equations can have zero solutions, one solution, or two solutions.
  o Students may be confused about the solution set for a system on linear inequalities.
  o Students may not consider points on the boundary lines as potential solutions, or might assume that they always are solutions to a system of linear inequalities.

- Algebraically Solving Systems of Equations
  o Students find one variable and do not continue to solve for the rest.
  o Students may not realize that when they arrive at a true statement with no variables, like 0 = 0, this indicates that there are infinite solutions to the system.
  o Students may not realize that when they arrive at an untrue statement with no variables, like 3 = 5, this indicates that there are no solutions to the system.
  o Students might substitute the value of one variable back into a wrong equation to try to solve for the other variable(s).
  o Students may try to add up the variables in the equations instead of canceling them out when performing elimination.
Students forget to multiply all terms of an equation when using the elimination method.
Students might not have all equations in the same form when attempting the elimination method.

- Modeling with Systems
  - Students neglect the real-world context when giving solutions for a system of equations that models a real-world problem.

**Classroom Challenge**

Consider this system of linear equations.

\[ y = ax + b \]
\[ y = cx + d \]

Part I:
Find values of \( a, b, c, \) and \( d \) that create the system of equations such that the following criteria hold true:

- The point \((-2, 4)\) is a solution to \( y = ax + b \), but is not a solution to \( y = cx + d \).
- The point \((4, 1)\) is the solution to the system of linear equations.
- The lines \( y = ax + b \) and \( y = cx + d \) are perpendicular to each other.

Part II:
Graph the system of linear equations. Explain how the graph shows that each of the criteria listed holds true.

**Possible solution pathway:**

Part I:
The points \((-2, 4)\) and \((4, 1)\) are solutions to \( y = ax + b \), so we can use the points to find the slope of this line: \( m = \frac{4 - 1}{-2 - 4} = \frac{3}{-6} = -\frac{1}{2} \). Next, we can substitute in the slope and one of the points to find the value of \( b \).

\[ y = ax + b \]
\[ 1 = -\frac{1}{2} (4) + b \]
\[ 3 = b \]

So the first equation of the system is \( y = -\frac{1}{2} x + 3 \).
We can use the fact that the lines are perpendicular to determine the slope for the line $y = cx + d$. Lines that are perpendicular have opposite reciprocal slopes, so the slope of this line will be 2. Next, we can use this slope and the point (4, 1) to determine the value of $b$.

\[
y = cx + d
\]

\[
1 = 2(4) + b
\]

\[
-7 = b
\]

So the second equation of the system is $y = 2x - 7$.

That means that $a = -\frac{1}{2}$, $b = 3$, $c = 2$, and $d = -7$.

Part II:

The graph of the system of linear equations is shown below. You can see that the point (–2, 4) lies on the graph of the first equation, but not on the second. The point (4, 1) lies on both graphs, so it is the solution to the system of equations. The lines do appear to be perpendicular.
Teacher Notes:

For students that struggle to get started, have them graph the points given in the criteria and ask them to think about what the graphs of the lines might look like. Students may need to be reminded that the slopes of perpendicular lines are opposite reciprocals.

For advanced students, have them create a real-world scenario that could be modeled by this system of linear equations. Ask them to interpret the meaning of the solution in context of the real-world scenario.

UNIT 5: POLYNOMIALS: PART ONE

Estimated Unit Time: approx. 11 Class Periods

Using their knowledge of polynomial multiplication, inverse operations, and the structure of expressions, students write polynomials in completely factored form (MP7). Students learn how to use long division to divide polynomials. They use the binomial theorem (MP6) to expand polynomials. They graph polynomial functions and analyze the graphs for key features. Students learn about synthetic division for polynomials and use it to apply the remainder theorem. In the final lesson of the unit, students identify possible rational roots of a polynomial function using the rational roots theorem. Then they relate multiplicity of a polynomial to the number of zeroes and how that appears on a graph.

Unit 5 Focus Standards

The following focus standards are intended to guide teachers to be purposeful and strategic in both what to include and what to exclude when teaching this unit. Although each unit emphasizes certain standards, students are exposed to a number of key ideas in each unit, and as with every rich classroom learning experience, these standards are revisited throughout the course to ensure that students master the concepts with an ever-increasing level of rigor.

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<tr>
<td>Use the structure of an expression to identify ways to rewrite it. For example, see $x^2 - y^2$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.</td>
<td></td>
</tr>
<tr>
<td>A2:A-SSE.A.2</td>
<td></td>
</tr>
<tr>
<td>Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x) + r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.</td>
<td></td>
</tr>
<tr>
<td>A2:A-APR.D.6</td>
<td></td>
</tr>
<tr>
<td>Use polynomial identities to describe numerical relationships. For example, the polynomial identity $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ can be used to generate Pythagorean triples.</td>
<td></td>
</tr>
<tr>
<td>A2:A-APR.C.4</td>
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</tr>
<tr>
<td>Standard Text</td>
<td>Standard ID</td>
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<tr>
<td>-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------</td>
<td>---------------------</td>
</tr>
<tr>
<td>Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.</td>
<td>A2:F-IF.C.7.c</td>
</tr>
<tr>
<td>For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.</td>
<td>A2:F-IF.B.4</td>
</tr>
<tr>
<td>Identify the effect on the graph of replacing ( f(x) ) by ( f(x) + k, ) ( kf(x), ) ( f(kx), ) and ( f(x + k) ) for specific values of ( k ) (both positive and negative); find the value of ( k ) given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.</td>
<td>A2:F-BF.B.3</td>
</tr>
<tr>
<td>Know and apply the Remainder Theorem: For a polynomial ( p(x) ) and a number ( a, ) the remainder on division by ( x – a ) is ( p(a), ) so ( p(a) = 0 ) if and only if ( (x – a) ) is a factor of ( p(x) ).</td>
<td>A2:A-APR.B.2</td>
</tr>
<tr>
<td>Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.</td>
<td>A2:A-APR.B.3</td>
</tr>
</tbody>
</table>
# Unit 5 Pacing Guide

<table>
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<th>Objectives</th>
<th>Standards</th>
<th>Number of Days</th>
</tr>
</thead>
<tbody>
<tr>
<td>Factoring Polynomials</td>
<td>• Analyze polynomial expressions to factor them completely.</td>
<td>A2:A-SSE.A.2</td>
<td>1.5</td>
</tr>
<tr>
<td>Completely</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Division of Polynomials</td>
<td>• Use long division to find quotients of polynomials.</td>
<td>A2:A-APR.D.6</td>
<td>1.5</td>
</tr>
<tr>
<td></td>
<td>• Use inverse operations to check the result of polynomial division.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>The Binomial Theorem</td>
<td>• Use the binomial theorem to expand binomials.</td>
<td>A2:A-APR.C.4</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>• Use the binomial theorem to find a specific term in an expansion.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Monomial Functions</td>
<td>• Analyze the key attributes of monomial functions.</td>
<td>A2:F-IF.C.7.c</td>
<td>1</td>
</tr>
<tr>
<td>Graphs of Polynomial</td>
<td>• Identify the key features of a polynomial function from a given graph.</td>
<td>A2:F-IF.B.4</td>
<td>1</td>
</tr>
<tr>
<td>Functions</td>
<td>• Describe the key features of a polynomial function.</td>
<td>A2:F-BF.B.3</td>
<td></td>
</tr>
<tr>
<td>Synthetic Division and</td>
<td>• Use synthetic division to divide a polynomial by a linear factor.</td>
<td>A2:A-APR.B.2</td>
<td>2</td>
</tr>
<tr>
<td>the Remainder Theorem</td>
<td>• Apply the remainder theorem.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>The Rational Roots</td>
<td>• Use the rational root theorem to determine possible roots of a polynomial function.</td>
<td>A2:A-APR.B.3</td>
<td>2</td>
</tr>
<tr>
<td>Theorem</td>
<td>• Determine the roots of and factor a polynomial function.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Unit Test</td>
<td></td>
<td></td>
<td>1</td>
</tr>
</tbody>
</table>

## Discussion Questions & Answers

1. Describe how to analyze the graph of a polynomial function.
   a. **Begin by looking at the end behavior of the graph.** If the graph has ends that open in the same direction, the function is even. If the graph has ends that open in opposite directions, then the function is odd. You can locate local maxima and local minima by studying the turning points of the graph, or where the graph changes from increasing to decreasing or vice versa. The maximum number of turning points is one less than the degree of the function. The x-intercept(s) can be estimated from the graph as well.

2. What is the remainder theorem, and how is it applied?
   a. **The remainder theorem says that when you divide a polynomial** \( f(x) \) **by** \( x - k \), **the remainder is** \( f(k) \). **It also says that if** \( f(k) = 0 \) **, then** \( x - k \) **is a factor of the polynomial** \( f(x) \) **and** \( k \) **is a root of** \( f(x) \). **You can use the remainder theorem and synthetic division to find**
the remainder when the polynomial is divided by a factor. You can determine whether \( x - k \) is a factor of \( f(x) \), or if \( k \) is a root of \( f(x) \). The theorem allows you to identify factors and roots of a polynomial function.

3. Can you use the rational roots theorem to identify all the zeros of a polynomial function? Why or why not?
   a. No, the rational roots theorem will only produce a list of possible rational zeros. You have to test each potential zero to verify if it is an actual zero of the function or not. A polynomial function can also have zeros that are irrational or complex, which the rational roots theorem cannot identify.

Common Misconceptions

- Factoring Polynomials
  - Students may try to distribute exponents incorrectly, i.e., \((a + b)^2 = a^2 + b^2\).
  - Students incorrectly apply exponent laws (e.g., multiply exponents instead of adding them) due to a lack of understanding where the laws come from.
  - Students might not recognize and factor out greatest common factors that contain variables.

- Polynomial Division
  - Students do not write the dividend in standard form before attempting to divide.
  - Students might forget to insert a placeholder term in the dividend.
  - Students may only apply subtraction to the first term, not the entire expression, in the long division algorithm.

- Binomial Theorem
  - Students incorrectly replace \( a \) and \( b \) with the actual binomial terms.
  - Students might misidentify the coefficients from Pascal’s triangle.

- Graphs of Polynomial Functions
  - Students may forget to rewrite a polynomial in standard form before analyzing the degree and leading coefficient to determine if it is even or odd.
  - Students might have difficulty interpreting the graph of a polynomial function that models a real-world problem.

- Remainder Theorem and Rational Root Theorem
  - Students may confuse the factor \( x - k \) and the root \( k \), which could lead to errors when using the remainder theorem and synthetic division.
  - Students incorrectly identify \( \frac{p}{q} \) when applying the rational roots theorem.
  - Students believe that the rational roots theorem lists all possible zeros, and not realize that there could be irrational and/or complex roots as well.

Classroom Challenge

A student wants to divide the polynomial \( 6x^3 + 20x^2 + 8x + 6 \) by \( 2x + 6 \). The student knows that synthetic division works when the divisor is of the form \( x - k \). To use synthetic division when the form of the divisor is \( ax - k \), the student figures it makes sense to use \(-3\) for \( k \) instead of \(-6\) because \(-6 \div 2 = -3\).
Part I: Determine the quotient using synthetic division if \( k = -3 \).

Part II: Divide the cubic polynomial by the original linear binomial using long division. How does the result of the long division compare to the result of the synthetic division?

Generalize: Can synthetic division be applied using \( \frac{k}{a} \) as the divisor when the divisor is of the form \( ax - k \)? Why or why not?

**Possible solution pathway:**

**Part I:**

\[
\begin{array}{c|cccc}
-3 & 6 & 20 & 8 & 6 \\
 & -18 & -6 & -6 & \\
\hline
 & 6 & 2 & 2 & 0
\end{array}
\]

The result, using synthetic division if \( k = -3 \) is \( 6x^2 + 2x + 2 \).

**Part II:**

\[
\begin{array}{cccccc}
2x + 6 & | & 6x^3 & + & 20x^2 & + 8x + 6 \\
 & - (6x^3 + 18x^2) & \\
\hline
 & 2x^2 & + & 8x & + 6 & \\
 & - (2x^2 + 6x) & \\
\hline
 & 2x & + & 6 & \\
 & - (2x + 6) & \\
\hline
 & 0 & \\
\end{array}
\]

The quotient using long division is \( 3x^2 + 1x + 1 \).

Using long division, every coefficient and constant in the quotient is one-half of the value of the corresponding term in the quotient of the synthetic division.

\( \frac{k}{a} \) cannot be used in synthetic division when the expression is in the form \( ax - k \).

The constant term in the divisor cannot just be divided by the coefficient of \( x \); it must be factored out. In this problem, that means \( 2x + 6 = 2(x + 3) \). This means that dividing the original dividend, \( 6x^3 + 20x^2 + 8x + 6 \), by \( 2x + 6 \) is the same as dividing the original dividend by \( 2(x + 3) \).

\[
\frac{6x^3+20x^2+8x+6}{2x+6} = \frac{6x^3+20x^2+8x+6}{2(x+3)}
\]

Using synthetic division with \( k = -3 \), we have a quotient of \( \frac{6x^2+2x+2}{2} \). Now each term is divided by 2, resulting in \( 3x^2 + 1x + 1 \). This is the same as the quotient from using long division.
Teacher Notes:

Students who struggle may benefit from visual representations of the division (i.e., area models) or may benefit from first working through a division problem with fewer terms.

Students who are in need of enrichment may benefit from determining the relationship between the linear coefficient in the divisor and the synthetic division solution.

Ex.

\[
\frac{6x^3 + 20x^2 + 8x + 6}{2x + 6} = \frac{6x^3 + 20x^2 + 8x + 6}{2(x + 3)} = \frac{1}{2} \cdot \frac{6x^3 + 20x^2 + 8x + 6}{x + 3}
\]

At this point, synthetic division can be used to divide \(6x^3 + 20x^2 + 8x + 6\) by \(x + 3\). The quotient would then be multiplied by one-half.

**UNIT 6: POLYNOMIALS: PART TWO**

*Estimated Unit Time: approx. 13 Class Periods*

This unit begins with the Fundamental Theorem of Algebra, giving students another tool for finding the roots for a polynomial function. Together with the work from the previous unit, students find roots using different methods, including using the remainder theorem. Students write polynomial functions given real and complex roots. They study functions that are quadratic in form and use their structure to rewrite them (MP7). Once rewritten, students use a variety of methods for solving the quadratic equations and identifying the zeros of the related function. From there, students graph polynomial functions from key features, including the zeroes of the function. In the final lesson in this unit, students transform polynomial equations into a system of equations. Then they use the graphing calculator to graph the system and identify solutions (MP5).

**Unit 6 Focus Standards**

The following focus standards are intended to guide teachers to be purposeful and strategic in both what to include and what to exclude when teaching this unit. Although each unit emphasizes certain standards, students are exposed to a number of key ideas in each unit, and as with every rich classroom learning experience, these standards are revisited throughout the course to ensure that students master the concepts with an ever-increasing level of rigor.

In the following table, green highlights indicate major work of the grade, blue highlights indicate supporting work, and yellow highlights indicate additional work.

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<td>A2:A-APR.B.2</td>
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</table>
Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.  

Use the structure of an expression to identify ways to rewrite it. For example, see \( x^4 - y^4 \) as \((x^2)^2 - (y^2)^2\), thus recognizing it as a difference of squares that can be factored as \((x^2 - y^2)(x^2 + y^2)\).  

Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.  

Explain why the \( x\)-coordinates of the points where the graphs of the equations \( y = f(x) \) and \( y = g(x) \) intersect are the solutions of the equation \( f(x) = g(x) \); find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where \( f(x) \) and/or \( g(x) \) are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.

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<tbody>
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<td><strong>Lesson</strong></td>
</tr>
</tbody>
</table>
| The Fundamental Theorem of Algebra | • Apply the fundamental theorem of algebra to determine the number of roots of a polynomial function.  
• Use the complex conjugate theorem to factor and solve polynomial equations. | A2:A-APR.B.3  
A2:A-APR.B.2 | 2 |
| Writing Polynomial Functions from Real Roots | • Write polynomial functions from real roots. | A2:A-APR.B.2 | 2 |
| Writing Polynomial Functions from Complex Roots | • Write polynomial functions from complex roots. | A2:A-APR.B.2 | 2 |
| Quadratic in Form Polynomials | • Identify fourth-degree equations that are quadratic in form and use an appropriate \( u\)-substitution.  
• Solve fourth-degree equations that are quadratic in form. | A2:A-SSE.A.2  
A2:A-APR.B.3  
A2:A-APR.B.2 | 2 |
| Graphing Polynomial Functions | • Graph polynomial functions using key features. | A2:A-APR.B.3  
A2:F-IF.C.7.c | 2 |
Lesson | Objectives | Standards | Number of Days
--- | --- | --- | ---
Solving Polynomial Equations using Technology | • Use technology to solve or approximate solutions of one-variable polynomial equations. | A2:A-REI.D.11 | 2

Unit Test | | | 1

Discussion Questions & Answers
1. Explain and justify the steps you would take to find the roots of the polynomial function $f(x) = x^4 - 2x^3 - x^2 - 4x - 6$.
   a. The degree of the polynomial is 4, so by the Fundamental Theorem of Algebra, we know we are looking for four roots. We can use the rational roots theorem to list the possible rational roots: ±1, ±2, ±3, ±6. Then we can use synthetic division to test these values, and if the remainder is 0 when we divide, then the remainder theorem states that the value is a root. Using this procedure, we identify two roots as 3 and −1. We have divided these out of the polynomial and are left with $x^2 + 2$. We use the zero product property and solve $x^2 + 2 = 0$ to identify the other roots. We can solve this using the square root property, completing the square, or the quadratic formula. Once we apply one of these methods, we find the other two roots: $i\sqrt{2}$ and $-i\sqrt{2}$. This makes sense that they are conjugates because the complex conjugate theorem states that any complex roots will exist in conjugate pairs.

2. How can you use the zeros of a polynomial function to sketch a graph of that function?
   a. The zeros of the function are the x-intercepts, so you can plot the intercepts from the zeros. Next, you can look at the degree and leading coefficient of the function to identify if it is even or odd which will determine the end behavior. You can then test values over each interval created by the x-intercepts to determine if the function is positive or negative over each interval.

3. Discuss how to solve a polynomial equation using a graphing calculator.
   a. To solve a polynomial equation using a graphing calculator, write the equation as a system of equations. Use the graphing calculator to graph the system, and then identify the point(s) of intersection(s) of the graphs. The x-coordinate of any intersection point is a solution to the related polynomial equation.

Common Misconceptions
• Polynomial Functions
  o Students may forget to rewrite a polynomial in standard form before determining its degree.
  o Students might think that if the graph of polynomial function does not have any x-intercepts, it does not have any roots.
  o Students may forget that complex and irrational roots come in conjugate pairs.
o Students may confuse the factor \( x - k \) and the root \( k \) when writing or graphing a polynomial function.

o Students might not recognize when a polynomial is quadratic in form.

- **Graphing Polynomial Functions**
  
  o Students may forget to rewrite a polynomial in standard form before analyzing the degree and leading coefficient to determine if it is even or odd.
  
  o Students do not consider the multiplicity of a zero to determine if a graph crosses or touches at the \( x \)-intercept.
  
  o Students incorrectly use the zeros or \( y \)-values when testing intervals to determine whether the function is positive or negative.

- **Polynomial Equations**
  
  o Students might not realize a point must lie on both graphs to be a solution to the system.
  
  o Students may identify the \( y \)-value of the intersection point as the solution.
  
  o Students neglect the real-world context when giving solutions for a system of equations that models a real-world problem.

**Classroom Challenge**

Write a standard form equation for a polynomial that intersects the \( x \)-axis at exactly two distinct points for each of the following descriptions:

1. The equation is quadratic.

2. The degree of the equation is odd.

3. The degree of the equation is even, but not 2.

For each equation, sketch a graph. Using the graphs, explain how the number of \( x \)-intercepts was not enough information to determine the exact degree of the polynomial equation.
Possible solution pathway:

1. The quadratic equation \( y = x^2 - 6 \) intersects the x-axis at two points.

2. The equation \( y = x^3 + 4x^2 - 3x - 18 \) is an odd function that has two distinct x-intercepts.
3. The equation $y = x^4 + 2x^3 - 11x^2 - 12x + 36$ is an even function with a degree of 4 that has two distinct $x$-intercepts.

The number of $x$-intercepts tells you the minimum degree of the polynomial equation. It cannot tell you if there is multiplicity at any of the $x$-intercepts. In the cubic graph, the left-most $x$-intercept has a multiplicity of two. In the fourth-degree graph, both $x$-intercepts have a multiplicity of two.

**Teacher Notes:**

For students who struggle, it could be helpful to have them start with the sketch of a graph of a function with the desired degree. Using translations (tracing paper, transparency sheets, sheet protectors, etc.), slide the graph of the function around the coordinate grid until the number of $x$-intercepts changes. Have students generalize what happened to the graph of the function and have them write the transformation algebraically.

Students may also benefit from writing the factored form of the equation before writing the standard form equation. The multiplicity should be easier to see when the equation is in factored form.

Students who need an additional challenge could be asked to use complex roots instead of $x$-intercepts when writing their equations and graphing. For example, “A polynomial has a pair of complex roots and an integer root. Write an odd degree polynomial equation that could represent the polynomial and graph the equation.”
UNIT 7: RATIONAL FUNCTIONS

Estimated Unit Time: approx. 16 Class Periods total

In this unit, students simplify and perform operations with rational expressions using the structure to identify ways to rewrite the expressions (MP7). Students solve rational equations and determine extraneous solutions. Then students move on to graph rational functions, using the structure of the rational functions to help determine holes in the graph and vertical and horizontal asymptotes (MP7). Rational functions are used to model and solve real-world problems (MP4).

Unit 7 Focus Standards

The following focus standards are intended to guide teachers to be purposeful and strategic in both what to include and what to exclude when teaching this unit. Although each unit emphasizes certain standards, students are exposed to a number of key ideas in each unit, and as with every rich classroom learning experience, these standards are revisited throughout the course to ensure that students master the concepts with an ever-increasing level of rigor.

In the following table, green highlights indicate major work of the grade, blue highlights indicate supporting work, and yellow highlights indicate additional work.

<table>
<thead>
<tr>
<th>Standard Text</th>
<th>Standard ID</th>
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</thead>
<tbody>
<tr>
<td>Use the structure of an expression to identify ways to rewrite it. For example, see ( x^4 - y^4 ) as ((x^2)^2 - (y^2)^2), thus recognizing it as a difference of squares that can be factored as ((x^2 - y^2)(x^2 + y^2)).</td>
<td>A2:A-SSE.A.2</td>
</tr>
<tr>
<td>Rewrite simple rational expressions in different forms; write ( \frac{a(x)}{b(x)} ) in the form ( q(x) + \frac{r(x)}{b(x)} ), where ( a(x) ), ( b(x) ), ( q(x) ), and ( r(x) ) are polynomials with the degree of ( r(x) ) less than the degree of ( b(x) ), using inspection, long division, or, for the more complicated examples, a computer algebra system.</td>
<td>A2:A-APR.D.6</td>
</tr>
<tr>
<td>Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.</td>
<td>A2:A-REI.A.2</td>
</tr>
<tr>
<td>Explain why the ( x )-coordinates of the points where the graphs of the equations ( y = f(x) ) and ( y = g(x) ) intersect are the solutions of the equation ( f(x) = g(x) ); find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where ( f(x) ) and/or ( g(x) ) are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.</td>
<td>A2:A-REI.D.11</td>
</tr>
<tr>
<td>Identify the effect on the graph of replacing ( f(x) ) by ( f(x) + k ), ( k f(x) ), ( f(kx) ), and ( f(x + k) ) for specific values of ( k ) (both positive and negative); find the value of ( k ) given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.</td>
<td>A2:F-BF.B.3</td>
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</tbody>
</table>
**Standard Text**

<table>
<thead>
<tr>
<th>Standard Text</th>
<th>Standard ID</th>
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</thead>
<tbody>
<tr>
<td>Determine an explicit expression, a recursive process, or steps for calculation from a context.</td>
<td>A2:F-BF.A.1.a</td>
</tr>
<tr>
<td>Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions.</td>
<td>A2:A-CED.A.1</td>
</tr>
</tbody>
</table>

**Unit 7 Pacing Guide**

<table>
<thead>
<tr>
<th>Lesson</th>
<th>Objectives</th>
<th>Standards</th>
<th>Number of Days</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simplifying Rational Expressions</td>
<td>• Simplify rational expressions using laws of integer exponents.</td>
<td>A2:A-APR.D.6</td>
<td>1</td>
</tr>
<tr>
<td>Simplifying Rational Expressions by Factoring</td>
<td>• Determine excluded values of rational expressions.</td>
<td>A2:A-SSE.A.2</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>• Simplify rational expressions using factoring techniques.</td>
<td>A2:A-APR.D.6</td>
<td></td>
</tr>
<tr>
<td>Multiplying and Dividing Rational Expressions</td>
<td>• Perform multiplication and division of rational expressions.</td>
<td>A2:A-SSE.A.2</td>
<td>2</td>
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<tr>
<td></td>
<td></td>
<td>A2:A-APR.D.6</td>
<td></td>
</tr>
<tr>
<td>Adding and Subtracting Rational Expressions</td>
<td>• Perform addition and subtraction of rational expressions.</td>
<td>A2:A-SSE.A.2</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>• Simplify complex rational expressions containing sums or differences.</td>
<td>A2:A-APR.D.6</td>
<td></td>
</tr>
<tr>
<td>Rational Equations</td>
<td>• Solve rational equations and determine extraneous solutions.</td>
<td>A2:A-REI.A.2</td>
<td>2</td>
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<tr>
<td></td>
<td>• Use rational equations to model and solve real-world problems.</td>
<td>A2:A-REI.D.11</td>
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<td></td>
<td>• Determine the reasonableness of a solution to a rational equation.</td>
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<tr>
<td>Vertical Asymptotes of Rational Functions</td>
<td>• Determine the vertical asymptotes and holes in the graph of a rational function having the x-axis as its only horizontal asymptote.</td>
<td>A2:A-SSE.A.2</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>• Solve problems involving inverse variation.</td>
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</tr>
<tr>
<td>Lesson</td>
<td>Objectives</td>
<td>Standards</td>
<td>Number of Days</td>
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</tr>
<tr>
<td>Graphing Rational Functions</td>
<td>• Determine the horizontal asymptotes of a rational function.</td>
<td>A2:A-SSE.A.2</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>• Graph rational functions that have only vertical or horizontal asymptotes.</td>
<td>A2:F-BF.B.3</td>
<td></td>
</tr>
<tr>
<td>Modeling with Rational Functions</td>
<td>• Model and solve real-world problems using rational functions.</td>
<td>A2:F-BF.A.1.a</td>
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<td>A2:A-CED.A.1</td>
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<td>Unit Test</td>
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</tbody>
</table>

**Discussion Questions & Answers**

1. How do you simplify a rational expression? Make sure to include how you know when a rational expression is simplified completely.
   a. **A rational expression is completely simplified when the numerator and denominator have no common factors and there are no negative exponents. You can factor out and divide common factors of the numerator and denominator to simplify the expression. You can also apply properties of exponents to simplify a rational expression.**

2. What is an extraneous solution to a rational equation? How do you identify an extraneous solution?
   a. **An extraneous solution is a solution to an equation, derived from an original equation that is not a solution of the original equation. To identify if solutions are extraneous, plug them back into the original equation and see if it holds true. If it does not hold true, the solution is extraneous.**

3. How do you use a rational function to identify vertical asymptotes and holes in the graphs of the function? Why do these locations specifically cause asymptotes and holes in the graphs?
   a. **To identify vertical asymptotes, you must simplify the function completely. Then the vertical asymptotes are located at the x-values that make the denominator equal 0. This is because you cannot divide by 0. To identify holes in the graph of a rational function, look for common factors of the numerator and denominator. The holes exist at the x-value(s) that make the common factor equal 0.**

**Common Misconceptions**

- Rational Expressions
  - Students incorrectly apply exponent laws (e.g., multiply exponents instead of adding them) due to a lack of understanding where the laws come from.
  - Students may miss a \(-1\) factor when clearing common factors (e.g., canceling \(x - 1\) and \(1 - x\)).
Students might cancel common terms instead of common factors (e.g., \( \frac{x+2}{x} = 2 \)).

Students may only partially divide (e.g., \( \frac{3x+6}{3} = x + 6 \)).

Students might make errors when distributing a negative.

Students incorrectly try to add or subtract without getting a common denominator first.

Students may think that excluded values are determined from the simplified expression instead of the original expression.

- **Rational Equations**
  - Students might forget to check for extraneous solutions.
  - Students may not multiply each term in the equation by the least common denominator to eliminate denominators.
  - Students incorrectly try to cross-multiply to solve a rational equation when the equation is not set up as a proportion.

- **Rational Function Graphs**
  - Students may incorrectly find the vertical asymptotes of a rational function by finding where the numerator equals zero, instead of where the denominator equals zero.
  - Students shift functions incorrectly when a constant is added or subtracted from the input value, i.e., \( f(x + c) \) shifts left and not right or \( f(x - c) \) shifts right and not left.
  - Students neglect the real-world context when giving solutions for a rational function that models a real-world problem.

### Classroom Challenges

Determine integer values for \( a, b, c, \) and \( d \) to complete the equation. State the restrictions of the domain.

\[
\frac{x^2 + 7x + 10}{x^2 + 2x - 15} \div \frac{x^2 + ax + b}{x^2 + cx + d} = \frac{x + 4}{x - 3}
\]

**Possible Solution Pathway:**

Multiply by the reciprocal of the divisor.

\[
\frac{x^2 + 7x + 10}{x^2 + 2x - 15} \cdot \frac{x^2 + cx + d}{x^2 + ax + b} = \frac{x + 4}{x - 3}
\]

Factor the numerator and denominator of the dividend.

\[
\frac{(x + 5)(x + 2)}{(x + 5)(x - 3)} \cdot \frac{x^2 + cx + d}{x^2 + ax + b} = \frac{x + 4}{x - 3}
\]

Cancel like terms to simplify the dividend.

\[
\frac{(x + 5)(x + 2)}{(x + 5)(x - 3)} \cdot \frac{x^2 + cx + d}{x^2 + ax + b} = \frac{x + 4}{x - 3}
\]
Determine what factors will multiply together in the numerator and denominator of the divisor to end up with a quotient of \( \frac{x+4}{x-3} \).

\[
\frac{(x+5)(x+2)}{(x+5)(x-3)} \cdot \left( \frac{?}{?} \right) = \frac{x+4}{x-3}
\]

To cancel to \((x+2)\) in the numerator of the dividend, I need a factor of \((x+2)\) in the denominator of the divisor. Since \((x+4)\) is the numerator of the quotient, I need a factor of \((x+4)\) in the numerator of the divisor.

\[
\frac{(x+5)(x+2)}{(x+5)(x-3)} \cdot \left( \frac{x+4}{?} \right) = \frac{x+4}{x-3}
\]

I now have the quotient I am looking for, but I need the numerator and denominator of the divisor to be quadratic. If I use the same factor in the numerator and denominator of the divisor, those factors will cancel out keeping the quotient \( \frac{x+4}{x-3} \).

\[
\frac{(x+5)(x+2)}{(x+5)(x-3)} \cdot \left( \frac{x+4}{x+2} \right) = \frac{x+4}{x-3}
\]

To determine a value for \(a, b, c,\) and \(d\), I can multiply each binomial together.

\[
\frac{x^2 + 7x + 10}{x^2 + 2x - 15} \cdot \frac{x^2 + 3x - 4}{x^2 + 1x - 2} = \frac{x+4}{x-3}
\]

\[
\frac{x^2 + 7x + 10}{x^2 + 2x - 15} \cdot \frac{x^2 + cx + d}{x^2 + ax + b} = \frac{x+4}{x-3}
\]

\[
a = 1, \ b = -2, \ c = 3, \text{ and } d = -4
\]

The restriction on the domain is \(x \neq -5, -4, -2, 1, 3\).

**Teacher Notes:**

If students struggle to get started, ask them to start by factoring.

It might also help to inform struggling students that the cross multiplication was completed correctly so those students do not spend their time on the operation itself.

Have students who need additional challenge graph the ratio function with the asymptotes and holes.
UNIT 8: RADICAL FUNCTIONS

Estimated Unit Time: approx. 21 Class Periods total

Students begin this unit by graphing the square root and cube root functions. They use the structure of these functions to relate transformations of the graphs to their parent function (MP7). From there, students simplify perfect and non-perfect roots. They convert between exponential expressions with rational exponents and radical form, and use this relationship to simplify, add, subtract, multiply, and divide expressions. Next, students solve radical equations, including determining extraneous solutions, and use radical equations to model real-world problems (MP4). A performance task ties it all together to conclude the unit, incorporating practice standards (MP1 and MP2) within the content standards covered in this project.

Unit 8 Focus Standards

The following focus standards are intended to guide teachers to be purposeful and strategic in both what to include and what to exclude when teaching this unit. Although each unit emphasizes certain standards, students are exposed to a number of key ideas in each unit, and, as with every rich classroom learning experience, these standards are revisited throughout the course to ensure that students master the concepts with an ever-increasing level of rigor.

In the following table, green highlights indicate major work of the grade, blue highlights indicate supporting work, and yellow highlights indicate additional work.

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</thead>
<tbody>
<tr>
<td>Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.</td>
<td>A2:F-IF.C.7.b</td>
</tr>
<tr>
<td>Identify the effect on the graph of replacing (f(x)) by (f(x) + k), (k f(x)), (f(kx)), and (f(x + k)) for specific values of (k) (both positive and negative); find the value of (k) given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.</td>
<td>A2:F-BF.B.3</td>
</tr>
<tr>
<td>Use the structure of an expression to identify ways to rewrite it. For example, see (x^4 - y^4) as ((x^2)^2 - (y^2)^2), thus recognizing it as a difference of squares that can be factored as ((x^2 - y^2)(x^2 + y^2)).</td>
<td>A2:A-SSE.A.2</td>
</tr>
<tr>
<td>Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. For example, we define (5^{1/3}) to be the cube root of 5 because we want ((5^{1/3})^3 = 5^{(1/3)\times3}) to hold, so ((5^{1/3})^3) must equal 5.</td>
<td>A2:N-RN.A.1</td>
</tr>
<tr>
<td>Rewrite expressions involving radicals and rational exponents using the properties of exponents.</td>
<td>A2:N-RN.A.2</td>
</tr>
</tbody>
</table>
### Standard Text

Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions.

### Standard ID

<table>
<thead>
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<tr>
<td>Solve simple rational and radical equations in one variable, and give examples</td>
<td>A2:A-REI.A.2</td>
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<tr>
<td>showing how extraneous solutions may arise.</td>
<td></td>
</tr>
<tr>
<td>Create equations and inequalities in one variable and use them to solve</td>
<td>A2:A-CED.A.1</td>
</tr>
<tr>
<td>problems. Include equations arising from linear and quadratic functions, and</td>
<td></td>
</tr>
<tr>
<td>simple rational and exponential functions.</td>
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</tbody>
</table>

### Unit 8 Pacing Guide

<table>
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<th>Objectives</th>
<th>Standards</th>
<th>Number of Days</th>
</tr>
</thead>
<tbody>
<tr>
<td>Graphing Radical Functions</td>
<td>• Relate transformations to the graphs of square root and cube root</td>
<td>A2:F-IF.C.7.b</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>and cube root functions to their parent function.</td>
<td>A2:F-BF.B.3</td>
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<tr>
<td></td>
<td>• Determine the domain and range of square root and cube root functions.</td>
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</tr>
<tr>
<td>Simplifying Perfect Roots</td>
<td>• Identify numbers and variable expressions that are perfect powers.</td>
<td>A2:A-SSE.A.2</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>• Simplify perfect roots.</td>
<td>A2:N-RN.A.1</td>
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<tr>
<td></td>
<td></td>
<td>A2:N-RN.A.2</td>
<td></td>
</tr>
<tr>
<td>Simplifying Nonperfect Roots</td>
<td>• Simplify nonperfect roots without rationalizing.</td>
<td>A2:A-SSE.A.2</td>
<td>2</td>
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<td>A2:N-RN.A.1</td>
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<td>A2:N-RN.A.2</td>
<td></td>
</tr>
<tr>
<td>Rational Exponents</td>
<td>• Evaluate numeric expressions using properties of rational exponents.</td>
<td>A2:N-RN.A.1</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>• Simplify algebraic expressions using properties of rational exponents.</td>
<td>A2:N-RN.A.2</td>
<td></td>
</tr>
<tr>
<td>Adding and Subtracting Radicals</td>
<td>• Identify like radicals.</td>
<td>A2:A-SSE.A.2</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>• Add and subtract radical expressions.</td>
<td>A2:N-RN.A.1</td>
<td></td>
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<td></td>
<td></td>
<td>A2:N-RN.A.2</td>
<td></td>
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<tr>
<td>Multiplying Radicals</td>
<td>• Perform multiplication of radical expressions.</td>
<td>A2:A-SSE.A.2</td>
<td>2</td>
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<tr>
<td></td>
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<td>A2:N-RN.A.1</td>
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<td>A2:N-RN.A.2</td>
<td></td>
</tr>
<tr>
<td>Dividing Radicals</td>
<td>• Perform division of radical expressions, rationalizing the</td>
<td>A2:N-REI.A.2</td>
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<tr>
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<td>denominator when necessary.</td>
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<tr>
<td>Lesson</td>
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<td>Standards</td>
<td>Number of Days</td>
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<td>-----------------------------------------------------------------------------</td>
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</tr>
<tr>
<td>Radical Equations and Extraneous Roots</td>
<td>• Model and solve mathematical and real-world problems using radical equations, and determine extraneous roots.</td>
<td>A2:N-REI.A.2</td>
<td>2</td>
</tr>
<tr>
<td>Solving Equations Containing Two Radicals</td>
<td>• Solve equations containing two radicals, and determine extraneous solutions.</td>
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<td>Performance Task: Roller Coaster Design</td>
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<td>A2:N-REI.A.2</td>
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<td>A2:A-CED.A.1</td>
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**Discussion Questions & Answers**

1. Under what circumstances does a simplified radical expression **not** require absolute value bars? Why?
   
   a. A simplified radical does not require absolute value bars if the domain restriction for the original expression is \( x \geq 0 \). With this domain restriction, the values under the radical are already restricted to include only those possible within the real number system. Additionally, a simplified radical does not require absolute value bars if the index of the radical is odd. If the index is odd, it is possible to have a negative (or positive) value underneath the radical and still remain within the real number system.

2. In an expression like \( \frac{x}{\sqrt{x} + 3} \), explain why the conjugate of the denominator, rather than the denominator itself, is used to rationalize the denominator.
   
   a. The conjugate of the denominator, in this case \( \sqrt{x} - 3 \), is used to rationalize the denominator because the resulting term or expression must contain no radicals. By multiplying by the conjugate, a difference of squares expression is created. When the expression is multiplied by its conjugate, the middle terms (the terms that would still contain the radical) cancel. If the expression itself were to be used, the middle term would be double the product of the first and last terms (in this case \( 2 \cdot 3\sqrt{x} \)) leaving a radical in the denominator.

3. Explain why the equation \( \sqrt{2 + x} = \sqrt{x - 3} \) must be squared twice in order to solve for possible values of \( x \), while the equations \( \sqrt{2 + x} = \sqrt{x} \) and \( \sqrt{2 + x} + 3 = x \) only need squared once.
   
   a. *In the first equation, when both sides are squared, the left side of the equation results in an expression with no radicals, but the right side of the expression still has a radical*
term. The sides must be squared a second time in order to get rid of the radical created by the initial multiplication. In the second equation, both radicals are isolated. When both sides are squared, both radicals will be eliminated. In the third equation, once the radical is isolated, one side contains a radical and the other side does not. When both sides are squared, neither side will have a radical in the result.

Common Misconceptions

- **Graphing Radical Functions**
  - Students may determine $\sqrt{x} = \pm \sqrt{x}$, resulting in a graph that appears to be a full parabola, opening horizontally with a range of all real numbers.
  - Students might treat the cube root function with the same domain and range restrictions as the square root function.
  - When transforming radical functions horizontally, students incorrectly consider $\sqrt{x} - h$ a translation to the left and $\sqrt{x} + h$ a translation to the right. Similarly, students consider $\sqrt{x} + k$ a downward translation and $\sqrt{x} - k$ an upward translation.
  - Students may use $h$ to determine vertical translations and $k$ to determine horizontal translations.
  - Students consider $y = -\sqrt{x}$ as a reflection over the $y$-axis instead of the $x$-axis.
  - Students might confuse $y = a\sqrt[3]{x}$ and $y = \sqrt[3]{ax}$.

- **Simplifying Perfect and Nonperfect Roots**
  - Students incorrectly halve instead of taking the square root, or divide by 3 instead of taking the cube root.
  - Students may rewrite $\sqrt[n]{x^y}$ as $\sqrt[n]{(x^a)^b}$ where $a + b = y$ instead of $ab = y$.
  - Students may apply $\sqrt[n]{x^m} = |x|$ for odd and even indices, or may neglect to apply the absolute value entirely, regardless of the index.

- **Rational Exponents**
  - Students might use the numerator as the index of the radical and the denominator as the power on the number under the radical (e.g., $a^{m/n} = \sqrt[n]{a^m}$).
  - When converting negative exponents to positive exponents, students negate the base rather than, or in addition to, the exponent when moving the power to the other part of the fraction.

- **Operations with Radicals**
  - Students may add or subtract unlike radicals.
  - Students might multiply only the first terms when distributing.
  - When multiplying using FOIL or raising an expression to a power, students multiply first term by first term and second term by second term, leaving no middle term.
Instead of multiplying by a fraction equal to 1, thus resulting in an equivalent fraction, students rationalize the denominator by multiplying by 1 over the existing denominator. (e.g., $\frac{a}{\sqrt{x}} \cdot \frac{1}{\sqrt{x}}$ resulting in $\frac{a}{x}$ which is not equivalent to the original fraction).

- Students rationalize the denominator of a fraction with an index greater than 2 by multiplying by the identical denominator (e.g., $\frac{a}{\sqrt[3]{x}} \cdot \frac{\sqrt[3]{x}}{\sqrt[3]{x}}$, incorrectly resulting in $\frac{a\sqrt[3]{x}}{x}$).

- When rationalizing a denominator that is an expression, students multiply by the same expression rather than the conjugate.

- Students might use the conjugate to rationalize the denominator of a fraction whose index is not 2.

### Radical Equations and Extraneous Roots

- Student may square binomials by distributing the exponent.
- Student might forget to check for extraneous solutions.
- Student incorrectly applies, or does not apply, the absolute value to even indices.
- Student applies the absolute value to the variable only, instead of the variable expression.
- Student attempts to solve before isolating the radical.
- Student may square or cube terms, rather than sides of the equation.

### Classroom Challenges

**Part I:** Algebraically determine the solution(s) to the equation $x - 3 = \sqrt{2x - 3}$.

**Part II:**
Graphically determine the solution(s) to the equation $x - 3 = \sqrt{2x - 3}$ by graphing each side of the equation as an individual function.

On one coordinate plane, graph each side of the equation as $y = x - 3$ and $y = \sqrt{2x - 3}$.

On a second coordinate plane, graph $y = (x - 3)^2$ and $y = (\sqrt{2x - 3})^2$.

Explain how the extraneous solutions to the original equation are represented on the two coordinate planes.

**Possible solution pathway:**

**Part I:**

\[ x - 3 = \sqrt{2x - 3} \]
\[ (x - 3)^2 = (\sqrt{2x - 3})^2 \]
\[ x^2 - 6x + 9 = 2x - 3 \]
\[ x^2 - 8x + 12 = 0 \]
\[ (x - 6)(x - 2) = 0 \]
\[ x = 2 \text{ or } 6 \]

Substituting into the original equation:

\[ 2 - 3 = \sqrt{2 \cdot 2 - 3} \]
\[ -1 = \sqrt{1} \]
\[ -1 \neq 1 \]

Since \(-1\) does not equal 1, \(x = 2\) is an extraneous solution.

\[ 6 - 3 = \sqrt{2 \cdot 6 - 3} \]
\[ 3 = \sqrt{9} \]
\[ 3 = 3 \]

This is true, so \(x = 6\) is a solution.

Part II:

The graphs of the original functions, as written in the equation:
The graphs of the squared functions, after the first step of solving for the unknown:

In the first graph, the solution to the equation is the intersection between the radical function and the line, the point (6, 3). There is no other point of intersection. This shows that the solution to the original equation is only (6, 3).

In the second graph, after the functions have been squared, there is now a linear function intersecting a quadratic function. There are two points of intersection between these functions: (2, 1) and (6, 3). The extraneous solution is the solution that exists \( x = 2 \) in the squared function, but not in the original function.

Teacher notes:

Students who need additional help may benefit from completing the activity with a simpler pair of equations (e.g., a square root expression set equal to a single variable or constant).

Students who struggle may also be presented with a contrasting equation and set of graphs in which there are two solutions for the variable, with no extraneous solutions. Comparing and contrasting the solutions and the graphs of the two equations may aid them in drawing conclusions.

Students who need extensions may benefit from completing a similar exercise, but with a cube root function. Students may be asked to generalize why square root functions, but not cube root functions, may contain extraneous solutions. Students may be asked to find a rule for the types of functions that do have extraneous solutions and explain why these functions have extraneous solutions and the other types of functions do not have extraneous solutions.
UNIT 9: EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Estimated Unit Time: approx. 17.5 Class Periods

In this unit, students identify exponential growth functions given in tables, graphs, and function rules. They model and solve real-world exponential growth problems using systems of equations (MP4). Students graph and determine the domain and range of exponential functions. Then they move on to solve exponential equations using properties of exponents. Next, students are introduced to logarithmic functions by applying their knowledge of inverse functions to exponential functions. Students use inverse relationships, properties, and algorithms to solve exponential and logarithmic equations. They model and solve a variety of real-world problems using exponential and logarithmic equations (MP4).

Unit 9 Focus Standards

The following focus standards are intended to guide teachers to be purposeful and strategic in both what to include and what to exclude when teaching this unit. Although each unit emphasizes certain standards, students are exposed to a number of key ideas in each unit, and, as with every rich classroom learning experience, these standards are revisited throughout the course to ensure that students master the concepts with an ever-increasing level of rigor.

In the following table, green highlights indicate major work of the grade, blue highlights indicate supporting work, and yellow highlights indicate additional work.

<table>
<thead>
<tr>
<th>Standard Text</th>
<th>Standard ID</th>
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<tr>
<td>Explain why the x-coordinates of the points where the graphs of the equations y = f(x) and y = g(x) intersect are the solutions of the equation f(x) = g(x); find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where f(x) and/or g(x) are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.</td>
<td>A2:A-REI.D.11</td>
</tr>
<tr>
<td>Use the properties of exponents to interpret expressions for exponential functions. For example, identify percent rate of change in functions such as y = (1.02)^t, y = (0.97)^t, y = (1.01)^{12t}, y = (1.2)^{10/10}, and classify them as representing exponential growth or decay.</td>
<td>A2:F-IF.C.8.b</td>
</tr>
<tr>
<td>Given a graph, a description of a relationship, or two input-output pairs (include reading these from a table), construct linear and exponential functions, including arithmetic and geometric sequences, to solve multistep problems.</td>
<td>A2:F-LE.A.2</td>
</tr>
<tr>
<td>Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.</td>
<td>A2:F-IF.C.7.e</td>
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<tr>
<td>Interpret the parameters in a linear, quadratic, or exponential function in terms of a context.</td>
<td>A2:F-LE.B.5</td>
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</table>
### Standard Text

Identify the effect on the graph of replacing \( f(x) \) by \( f(x) + k \), \( k f(x) \), \( f(kx) \), and \( f(x + k) \) for specific values of \( k \) (both positive and negative); find the value of \( k \) given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

Use the properties of exponents to transform expressions for exponential functions. For example, the expression \( 1.15^t \) can be rewritten as \( (1.15^{(1/12)})^{12t} \cdot 1.012^{12t} \) to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.

For exponential models, express as a logarithm the solution to \( a b^{(ct)} = d \) where \( a \), \( c \), and \( d \) are numbers and the base \( b \) is 2, 10, or \( e \); evaluate the logarithm using technology.

Solve an equation of the form \( f(x) = c \) for a simple function \( f \) that has an inverse and write an expression for the inverse. For example, \( f(x) = 2x^3 \) or \( f(x) = (x+1)/(x-1) \) for \( x \neq 1 \).

Determine an explicit expression, a recursive process, or steps for calculation from a context.

Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions.

### Standard ID

A2:F-BF.B.3

A2:A-SSE.B.3.c

A2:F-LE.A.4

A2:F-BF.B.4.a

A2:F-BF.A.1.a

A2:A-CED.A.1

### Unit 9 Pacing Guide

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<tr>
<th>Lesson</th>
<th>Objectives</th>
<th>Standards</th>
<th>Number of Days</th>
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</table>
| Exponential Growth Functions | • Identify an exponential growth function given tables, graphs, and function rules, determining the rate of change.  
• Graph an exponential growth function, and state the domain and range.  
• State the domain and range of an exponential growth function.  
• Write an exponential growth function to model a real-world problem, pointing out constraints in the modeling context. | A2:A-REI.D.11  
A2:F-IF.C.8.b  
A2:F-LE.A.2  
A2:F-IF.C.7.e  
A2:F-LE.B.5 | 1 |
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<td>Graphing Exponential Functions</td>
<td>• Identify exponential functions.</td>
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<td></td>
<td>• Determine the domain and range of exponential functions.</td>
<td>A2:F-BF.B.3</td>
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<td></td>
<td>• Graph exponential functions.</td>
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<tr>
<td>Solving Exponential Equations by Rewriting the Base</td>
<td>• Solve exponential equations by rewriting bases.</td>
<td>A2:A-SSE.B.3.c</td>
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<tr>
<td>Graphing Logarithmic Functions</td>
<td>• Identify logarithmic functions.</td>
<td>A2:F-IF.C.7.e</td>
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<td>• Determine the domain and range of logarithmic functions.</td>
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<td>• Identify and analyze the graphs of logarithmic functions.</td>
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<td>A2:F-BF.B.4.a</td>
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<td>Evaluating Logarithmic Expressions</td>
<td>• Evaluate logarithmic expressions by converting between logarithmic and exponential forms.</td>
<td>A2:F-LE.A.4</td>
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<td>• Solve logarithmic equations by converting between logarithmic and exponential forms.</td>
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<td></td>
<td>• Evaluate common logarithms using a calculator.</td>
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<tr>
<td>Solving Logarithmic Equations using Technology</td>
<td>• Rewrite logarithmic expressions using the change of base algorithm.</td>
<td>A2:A-REI.D.11</td>
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<td>• Solve a one-variable equation containing logarithms by transforming it into a system of equations.</td>
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<tr>
<td>Properties of Logarithms</td>
<td>• Evaluate, expand, and simplify logarithmic expressions using properties of logarithms.</td>
<td>A2:F-LE.A.4</td>
<td>1.5</td>
</tr>
<tr>
<td>Solving Equations using Properties of Logarithms</td>
<td>• Apply properties of logarithms to solve logarithmic equations.</td>
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<td>• Determine extraneous solutions of logarithmic equations.</td>
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<tr>
<td>Base e</td>
<td>• Apply properties of logarithms and exponents to solve exponential and logarithmic equations having base e.</td>
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<td>• Analyze exponential and logarithmic functions in base e to determine key features of the graph.</td>
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<td>• Determine the domain and range of exponential and logarithmic functions in base e.</td>
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Discussion Questions & Answers

1. An exponential function increases at a rate of 20% per unit of time. Four students write what they believe is the power of the associated exponential expression:

   Student A: 0.20^x
   Student B: 0.80^x
   Student C: 1.20^x
   Student D: 20^x

Differentiate between the four responses. Explain which student’s expression actually models the situation and what misconception each of the other students may have that resulted in their incorrect responses.

   a. **Student A is incorrect.** This student used the percent as a decimal as the base of the exponent, without considering the original value.
   b. **Student B is incorrect.** This student used 100% – 20% = 80%. You would use 80%, or 0.80, as the base of the exponent if the situation were to represent 20% decay. There would be 80% remaining after each unit of time.
   c. **Student C is correct.** A 20% increase is the same as taking the original 100% plus the extra 20%. Written as a decimal, 120% is 1.20.
   d. **Student D is incorrect.** This student used the given percent as the base of the exponent without converting to a decimal or considering that 100% of the original value is still present after the increase per unit time.

2. Explain how to solve the equation $36 \cdot 6^{-2x} = 36^{1-x}$.

   a. The equation can be written as powers with the same base. Each instance of 36 can be rewritten as $6^2$, resulting in the equation $(6)^2 \cdot 6^{-2x} = (6^2)^{1-x}$, or $6^2 \cdot 6^{-2x} = 6^{2-2x}$.
On the left side of the equation, the bases are the same, and the powers are being multiplied. The exponents can be combined using addition: $6^{2-2x} = 6^{2-2x}$. Because the bases are the same, the exponents must be equal: $2 - 2x = 2 - 2x$. Both sides of the equation are identical. This means that for any value of $x$, the expressions will be equivalent. This exponential equation has infinitely many solutions.

3. Show that the equation $\log(x - 10) - \log(x + 12) = \log x$ has no solution.

   a. The left side of the equation can be rewritten as $\log\left(\frac{x - 10}{x + 12}\right)$ using the quotient property.

   Both sides can be exponentiated, which results in the equation $\frac{x - 10}{x + 12} = x$. Next, both sides can be multiplied by $x + 12$. The result is $x - 10 = x(x + 12)$. Setting one side equal to zero, we have $0 = x^2 + 11x + 10$. This is factorable: $0 = (x + 10)(x + 1)$. Using the zero-product property, $x = -10, -1$. The argument in the first term on the left and the only term on the right will be negative using either possible value for $x$, making it so that neither is an actual solution to the equation. Graphically, $\log(x - 10) - \log(x + 12)$ and $\log x$ never intersect.

Common Misconceptions

- Exponential Growth Functions
  - Students incorrectly treat $b^x$ as $bx$.
  - Students may compute $b^0 = 0$ instead of $b^0 = 1$.
  - Instead of transforming bases with negative exponents by negating the exponent and moving the power to the other part of the fraction, students may negate the base and leave the expression otherwise as written (e.g., $b^{-3} = -b^3$ instead of $b^{-3} = \frac{1}{b^3}$).
  - Students identify the rate of change by subtraction rather than by division.
  - Students confuse the initial value and rate of change when writing the equation describing a function.
  - Students may determine a growth rate for a contextual problem without adding 1 to represent 100% of the previous, or initial, value.

- Graphing Exponential Functions
  - Students may determine a growth rate for a contextual problem without adding 1 to represent 100% of the previous, or initial, value. Similarly, students may determine a decay factor without subtracting the rate from 1.
  - Students confuse $b^x + k$ and $b^{x+k}$ or $b^{x+h}$ and $b^x + h$, resulting in horizontal shifts that should have been vertical shifts and vice versa.
  - Students incorrectly associate $b^{x-h}$ with horizontal motion to the left and $b^{x+h}$ with horizontal motion to the right.
  - Students become confused over which axis the graph is reflected when $a$ in $y = ab^x$ and $y = b^x$ are negated.
• **Solving Exponential Equations**
  - Students incorrectly use addition to simplify powers raised to powers and multiplication to combine the exponents of the product of two powers with the same base.
  - Students mistake no solution for infinite solutions, and vice versa. Alternatively, students mistake a solution of $x = 0$ for no solution.

• **Graphing Logarithmic Functions**
  - Students negate each coordinate, rather than switching the coordinates, when determining the inverse.
  - Students confuse the transformational result associated with a positive or negative $a$-value.
  - Students confuse the transformational result associated with $h$ and $k$.

• **Evaluating Logarithmic Expressions and Solving Logarithmic Equations**
  - Students incorrectly attempt arbitrary multiplicative relationships between values in a logarithm (e.g., $\log_{25} 5$ might be evaluated as 2 since 5 squared is 25, or $\log_{4} 8$ might be evaluated as 2 because 4 times 2 is 8).
  - Students may rewrite logarithmic equations as incorrect exponential equations.
  - Students do not understand how the domain restriction on a logarithm affects the possible solutions when solving equations.
  - Students might compute $\log_{b} c = \frac{\log c}{\log b}$ and then equate this to $\log_{b}^\frac{c}{b}$.
  - When solving a system using graphing, students confuse which variable is the actual solution.
  - Students believe that any negative value of $x$ is extraneous without substituting the value in to the original equation.
  - When using exponentiation to solve, students reverse the order of the base and the exponent.
  - Students may exponentiate terms rather than sides.

• **Properties of Logarithms**
  - Instead of applying the product or quotient property of logarithms, students find the values of the components and multiply or divide them (e.g., $\log(10) = \log(2 \times 5)$). If $\log2$ and $\log5$ are known, students may multiply those known values instead of adding them.
  - Students may confuse the quotient property of logarithms and the change of base formula.
  - Students may confuse power to a power with the power property of logarithms.
Classroom Challenge

i. What is the simplified value of the expression \((\log_39)(\log_9729)\)?

ii. What is the simplified value of the expression \((\log_216)(\log_{16}256)\)?

iii. Apply: Evaluate the expression \((\log_4x)(\log_x8)\) by setting each factor equal to a different variable and then solving for the unknown product \(yz\).

For example:
\((\log_4x)(\log_x8) = yz\)
\(\log_4x = y\)
\(\log_x8 = z\)

iv. Generalize: How does the product of the expressions relate to the values in the original expressions?

Possible solution pathway:

i. I know that \(\log_39 = 2\), because \(3^2 = 9\). I also know that \(\log_9729 = 3\), because \(9^3 = 729\).

The product of 2 and 3 is 6.

ii. I know that \(\log_216 = 4\), because \(2^4 = 16\). I also know that \(\log_{16}256 = 2\), because \(16^2 = 256\).

The product of 4 and 2 is 8.

iii. If \((\log_4x)(\log_x8) = yz\), then

\(\log_4x = y\)
\(\log_x8 = z\)

So \(4y = x\) and \(x^z = 8\).

Because I know what \(x\) is, I can substitute \(4y\) in for \(x\) in the second equation:

\((4y)^z = 8\) or \(4^{yz} = 8\).

I can rewrite both 4 and 8 as powers of 2: \(4 = 2^2\) and \(8 = 2^3\).

So: \((2^2)^y = 2^3\)

\(2^{2yz} = 2^3\), which means \(2yz = 3\) and \(yz = \frac{3}{2}\).

The product of the expressions is 1.5.
iv. I noticed a pattern in each pair of expressions. The base of the second logarithm matches a number in the first logarithm.

\((\log_3 9)(\log_9 729)\)

\((\log_2 16)(\log_{16} 256)\)

\((\log_4 x)(\log_x 8)\)

There is a direct relationship between the 3 and the 729 because \(3^6 = 729\). The power 3 is raised to is equal to the product of the logarithms.

The same relationship exists between the 2 and 256. \(2^8 = 256\). The power 2 is raised to is equal to the product of those logarithms.

In the final expression, \((\log_4 x)(\log_x 8)\), the product should be equal to \(4^n = 8\). This can be solved using logarithms \((\log_4 = 8)\), or by finding a common base, like I did in part iii.

Teacher notes:

Students who struggle may benefit from having the common value in each expression pointed out explicitly. The third expression can also be changed so that the values are directly related without having to find a common base (e.g. \((\log_4 x)(\log_4 16)\)).

It may be helpful to point out the formula: \((\log_a b)(\log_b c) = \log_c c\)

Students who may benefit from an extension can be asked to determine a relationship that exists for division: \(\frac{1}{\log_a b} = \log_b a\).

UNIT 10: TRIGONOMETRIC FUNCTIONS

Estimated Unit Time: approx. 10.5 Class Periods

In Geometry, students began trigonometry through a study of right triangles. In this unit, they extend the three basic functions to the entire unit circle (MP1). Students evaluate the six trigonometric functions for all angle measures using the unit circle, special right triangles, co-terminal angles, and technology. Students investigate the periodicity of the sine and cosine functions and use this knowledge to sketch their graphs. They use periodic functions to model and solve real-world problems (MP4).

Unit 10 Focus Standards

The following focus standards are intended to guide teachers to be purposeful and strategic in both what to include and what to exclude when teaching this unit. Although each unit emphasizes certain standards, students are exposed to a number of key ideas in each unit, and, as with every rich classroom learning experience, these standards are revisited throughout the course to ensure that students master the concepts with an ever-increasing level of rigor.
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<tr>
<td>Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.</td>
<td>A2:F-TF.A.1</td>
</tr>
<tr>
<td>Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.</td>
<td>A2:F-TF.A.2</td>
</tr>
<tr>
<td>Prove the Pythagorean identity ( \sin^2(\Theta) + \cos^2(\Theta) = 1 ) and use it to find ( \sin(\Theta) ), ( \cos(\Theta) ), or ( \tan(\Theta) ) given ( \sin(\Theta) ), ( \cos(\Theta) ), or ( \tan(\Theta) ) and the quadrant.</td>
<td>A2:F-TF.C.8</td>
</tr>
<tr>
<td>Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.</td>
<td>A2:F-IF.C.7.e</td>
</tr>
<tr>
<td>Identify the effect on the graph of replacing ( f(x) ) by ( f(x) + k ), ( k f(x) ), ( f(kx) ), and ( f(x + k) ) for specific values of ( k ) (both positive and negative); find the value of ( k ) given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.</td>
<td>A2:F-BF.B.3</td>
</tr>
<tr>
<td>For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.</td>
<td>A2:F-IF.B.4</td>
</tr>
<tr>
<td>Determine an explicit expression, a recursive process, or steps for calculation from a context.</td>
<td>A2:F-BF.A.1.a</td>
</tr>
<tr>
<td>Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.</td>
<td>A2:F-TF.B.5</td>
</tr>
</tbody>
</table>

**Unit 10 Pacing Guide**

<table>
<thead>
<tr>
<th>Lesson</th>
<th>Objectives</th>
<th>Standards</th>
<th>Number of Days</th>
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</thead>
<tbody>
<tr>
<td>Angles in Standard Position</td>
<td>• Identify characteristics of angles in standard position.</td>
<td>A2:F-TF.A.1</td>
<td>1</td>
</tr>
<tr>
<td>Lesson</td>
<td>Objectives</td>
<td>Standards</td>
<td>Number of Days</td>
</tr>
<tr>
<td>------------------------------</td>
<td>-----------------------------------------------------------------------------</td>
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</tr>
</tbody>
</table>
| Radian Measure               | • Convert between degree and radian measure.  
                                | • Use the definition of radian measure to calculate arc lengths, radii, and angle measures. | A2:F-TF.A.1        | 1              |
| The Unit Circle              | • Find the sine, cosine, and tangent values of angle measures using the unit circle.  
                                | • Compare sine, cosine, and tangent values for angles having the same reference angle. | A2:F-TF.A.2        | 1              |
| Reciprocal Trigonometric Functions | • Solve right triangle trigonometry problems involving reciprocal trigonometric functions.  
                                | • Simplify expressions involving the six trigonometric functions using reciprocal relationships.  
                                | • Evaluate the six trigonometric functions for special angles. | A2:F-TF.A.2        | 1              |
| Evaluating the Six Trigonometric Functions | • Evaluate the six trigonometric functions for angles in degrees or radians based on one or more given trigonometric function values.  
                                | • Evaluate the six trigonometric functions for angles in degrees or radians given a point on the terminal ray. | A2:F-TF.A.2, A2:F-TF.C.8 | 1              |
| Graphing Sine and Cosine     | • Analyze key features of sine and cosine functions from equations and graphs. | A2:F-IF.C.7.e, A2:F-BF.B.3 | 1.5            |
| Changes in Period and Phase Shift of Sine and Cosine Functions | • Analyze key features of sine and cosine functions from equations and graphs. | A2:F-IF.C.7.e, A2:F-BF.B.3 | 1.5            |
| Unit Test                    |                                                                             |                    | 1              |
Discussion Questions & Answers

1. Consider the unit circle. Under what circumstance, if any, are each of the 6 trigonometric functions undefined? Why? (Use $0 \leq \theta < 360$.)

   a. The sine function is defined as the $y$-value of a point on the unit circle. At all points on the unit circle, there is an associated $y$-value. Therefore, the sine function is defined for all points on the unit circle.

   b. The cosine function is defined as the $x$-value of a point on the unit circle. At all points on the unit circle, there is an associated $x$-value. Therefore, the cosine function is defined for all points on the unit circle.

   c. The tangent function is sine divided by cosine, so the tangent function is undefined at all points where the cosine value is 0. The cosine value is 0 where the $x$-value of the associated ordered pair is 0. These points are located along the $y$-axis. This means for $90^\circ$ and $270^\circ$ (or $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ radians), the tangent is undefined.

   d. The cosecant is the reciprocal of the sine value. Where the sine value is 0, along the $x$-axis, the cosecant value is undefined. This means at $0^\circ$ and $180^\circ$ (or 0 and $\pi$ radians), the cosecant is undefined.

   e. The secant is the reciprocal of the cosine value. Where the cosine value is 0, along the $y$-axis, the secant value is undefined. This means for $90^\circ$ and $270^\circ$ (or $\frac{\pi}{2}$ and $\frac{3\pi}{2}$), the secant is undefined.

   f. The cotangent is the reciprocal of the tangent value. Where the tangent has a value of 0 (where the sine equals zero), the cotangent will be undefined. The sine value is zero along the $x$-axis.

2. The graphs of $\sin \theta$ and $\cos \theta$ are the same except for a horizontal shift. Using the unit circle, explain why this is true.

   a. The graphs of $\sin \theta$ and $\cos \theta$ are plotted as points (angle measure, distance). Beginning at $0^\circ$, the $x$-axis, the cos value is 1 because the $x$-value is 1. Traveling counterclockwise, the graph of the cos value decreases because the distance between the $y$-axis and the perimeter of the circle decreases (the $x$-value). When the circle intersects the $y$-axis, the cosine value is 0. The angle measure at that point is $90^\circ$ or $\frac{\pi}{2}$ radians.

   The sine value relates to the distance along the $y$-axis instead of along the $x$-axis. So, starting at $0^\circ$, the sine value is 0, because this point on the circle is on the $x$-axis. Moving counterclockwise, the graph of the sine value increases because the distance between the $x$-axis and the perimeter of the circle (the $y$-value) increases. When the circle intersects the $y$-axis, the sine value is at its maximum value, 1. At that point the angle measure is $90^\circ$, or $\frac{\pi}{2}$ radians.
The maximum value for the sine function occurs at 90° and the maximum value for the cosine function occurs at 0°. They are separated by 90°, or $\frac{\pi}{2}$ radians. The minimum value for the sine function occurs at 0°, while the minimum value for the cosine function occurs at 90°. These are also separated by 90°, or $\frac{\pi}{2}$ radians. This pattern continues around the circle, with each maximum and minimum for the two functions separated by 90°.

3. For the function $y = a \sin(b(x - h)) + k$, describe how different values of $a$, $b$, $h$, and $k$ affect the appearance of the graphed function compared to the parent function.

   a. The $a$ value stands for the amplitude. If the $a$ value is greater than 0 and less than 1, then the transformed graph will appear vertically compressed. If the $a$ value is greater than 1, the graph will appear vertically stretched. If the $a$ value is negative, the graph will be reflected over the $x$-axis.

   b. The $b$ value is used to determine the number of cycles of the transformed function within the period of the parent function. If the $b$ value is between 0 and 1, the graph will appear horizontally stretched because there will be less than a full cycle in the same amount of horizontal space that it took the parent function to complete one full cycle. If the $b$ value is greater than 1, the graph will appear horizontally compressed because there will be more than one full cycle in the same space it took the parent function to complete one cycle.

   c. The $h$ value is a horizontal translation $h$ units. If the $h$-value is negative, the graph shifts left, if the $h$ value is positive, the graph shifts right. Note that a negative $h$ value will appear as $(x + h)$ and a positive $h$ value will appear as $(x - h)$.

   d. The $k$ value is a vertical translation. A positive $k$ value results in an upward shift, while a negative $k$ value results in a downward shift.

**Common Misconceptions**

- **Angles**
  - Students incorrectly state that a clockwise-opening angle is positive and a counterclockwise-opening angle is negative, or students entirely ignore the directionality of the angle when labeling the measure of the angle.
  - When converting between degrees and radians, students use the reciprocal of the correct conversion factor, lining up the units horizontally rather than diagonally.

- **Trigonometric Functions**
  - Students confuse the ratios that describe sine, cosine, and tangent.
  - Students confuse the axis associated with sine and cosine, and may also write tangent as the reciprocal of the slope.
  - Students incorrectly recall values associated with special right triangles.
Students may memorize reference angle formulas instead of learning to determine the reference angle by using the angle’s relationship to the x-axis.

Students may memorize rules for signs of trig function values instead of learning to determine the sign based on the placement of the terminal side of the angle.

Students might group secant and sine as well as cosecant and cosine.

Students may square the angle measure instead of the trig ratio.

- **Graphing Trigonometric Functions**
  - Students may have limited understanding of radians. The values on the x-axis may be unclear.
  - Students incorrectly measure amplitude as the distance from crest to trough.
  - Students confuse period and frequency.
  - Students confuse the direction of the transformations.
  - Students do not recognize the difference between \( \sin(b(x - h)) \) and \( \sin(bx - h) \).

**Classroom Challenge**

Write two separate real-world scenarios to model a relationship between two quantities:

1. using a sine function
2. using a cosine function

For each scenario, include an equation and corresponding graph that models the scenario.

**Possible solution pathway:**

1. A population of a type of animal fluctuates naturally. The estimated population at any time, \( t \), is predicted using a model that shows, from a stable position of 1,500 organisms, the population increases by 50 before it experiences a decline to 50 below the stable position. This cycle repeats itself every 10 years.

\[
p = a \sin(bt) + d \quad ; \quad (t, p), \text{ where } t = \text{time, in years and } p \text{ is population.}
\]

- \( a \) is the amplitude: 50.
- \( b \) is \( \frac{2\pi}{\text{period}} \). So, \( b = \frac{2\pi}{10} \text{ or } \frac{\pi}{5} \).
- \( d \) is the initial value and the midline: 1,500.
So, my equation is \( p = 50\sin\left(\frac{\pi}{5}t\right) + 1500. \)

2. A pendulum is pulled back so that the bottom of the pendulum is 4 inches from the resting point. Once released, the pendulum moves right and left, each time 4 inches from its lowest point. It takes the pendulum 2 seconds to complete one full right-left-right cycle.

\[ d = a\cos(bt); \ (t, \ d), \text{ where } t = \text{time, in seconds and } d \text{ is displacement, in inches.} \]

\( a \) is the amplitude: 4 inches.

\( b \) is \( \frac{2\pi}{\text{period}} \). The period is how long it takes to make one full cycle, in this case 2 seconds. So \( b = \frac{2\pi}{2} \) or just \( \pi \).

So, my equation is \( d = 4\cos(\pi t). \)
Teacher Notes:

Students who struggle may benefit from group brainstorming of situations that can be modeled using sine and cosine functions. Teachers may have to clarify the distinction between the two types of functions.

When writing the equation, students who struggle may benefit by simultaneously writing possible equations and seeing the corresponding graphs (e.g., watching how changing the numbers changes the graph of the function). It may help to analyze the possible graph with the students to see if the graph does effectively model the situation, and how the different components of the graph change as the components of the equations change.

Students may also benefit from seeing the equation/graph as a table of values.

Students who would benefit from extensions could be required to work with phase change and vertical shifts.

UNIT 11: MODELING AND ANALYSIS OF FUNCTIONS

Estimated Unit Time: approx. 16 Class Periods

In this unit, students extend and apply their knowledge of function families. They graph piecewise defined and step functions, interpreting key features of the graphs. Students combine functions using addition, subtraction, and multiplication. Then they compare properties of multiple functions represented in different ways. Students determine a function that best models a data set and interprets key features of the graphs of the models (MP4). Using a regression calculator, students determine and use linear, quadratic, and exponential models (MP4 and MP5). They analyze a function graphically, tabularly, and verbally to interpret key features, including rate of change. A performance task ties it all together to conclude the unit, incorporating practice standards (MP3 and MP6) within the content standards covered in this project.

Unit 11 Focus Standards

The following focus standards are intended to guide teachers to be purposeful and strategic in both what to include and what to exclude when teaching this unit. Although each unit emphasizes certain standards, students are exposed to a number of key ideas in each unit, and, as with every rich classroom learning experience, these standards are revisited throughout the course to ensure that students master the concepts with an ever-increasing level of rigor.

In the following table, green highlights indicate major work of the grade, blue highlights indicate supporting work, and yellow highlights indicate additional work.
<table>
<thead>
<tr>
<th>Standard Text</th>
<th>Standard ID</th>
</tr>
</thead>
<tbody>
<tr>
<td>For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.</td>
<td>A2:F-IF.B.4</td>
</tr>
<tr>
<td>Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.</td>
<td>A2:F-IF.C.7.b</td>
</tr>
<tr>
<td>Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions.</td>
<td>A2:A-CED.A.1</td>
</tr>
<tr>
<td>Combine standard function types using arithmetic operations. For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.</td>
<td>A2:F-BF.A.1.b</td>
</tr>
<tr>
<td>Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). For example, given a graph of one quadratic function and an algebraic expression for another, determine which has the larger maximum.</td>
<td>A2:F-IF.C.9</td>
</tr>
<tr>
<td>Use the properties of exponents to interpret expressions for exponential functions. For example, identify percent rate of change in functions such as ( y = (1.02)^t ), ( y = (0.97)^t ), ( y = (1.01)^{12t} ), ( y = (1.2)^{(t/10)} ), and classify them as representing exponential growth or decay.</td>
<td>A2:F-IF.C.8.b</td>
</tr>
<tr>
<td>Fit a function to the data; use functions fitted to data to solve problems in the context of the data. Use given functions or choose a function suggested by the context. Emphasize exponential models.</td>
<td>A2:S-ID.B.6.a</td>
</tr>
<tr>
<td>Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.</td>
<td>A2:F-IF.B.6</td>
</tr>
</tbody>
</table>

**Unit 11 Pacing Guide**

<table>
<thead>
<tr>
<th>Lesson</th>
<th>Objectives</th>
<th>Standards</th>
<th>Number of Days</th>
</tr>
</thead>
</table>
| **Piecewise Defined Functions** | • Graph piecewise defined functions.  
• Evaluate piecewise defined functions.  
• Determine the domain, range, and continuity of piecewise defined functions.                                                                 | A2:F-IF.B.4  
A2:F-IF.C.7.b  
A2:A-CED.A.1 | 1              |
<table>
<thead>
<tr>
<th>Lesson</th>
<th>Objectives</th>
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<th>Number of Days</th>
</tr>
</thead>
</table>
| Step Functions                 | • Evaluate step functions.  
• Analyze step functions to determine key features of the graph.  
• Use step functions to model real-world problems.                                                                                      | A2:F-IF.B.4, A2:F-IF.C.7.b             | 1              |
| Combining Functions            | • Construct a function that represents the combination of linear, exponential, or quadratic functions through addition or subtraction.  
• Construct a function that represents the combination of two linear functions through multiplication.                                   | A2:F-BF.A.1.b                          | 2              |
| Operations with Multiple Functions | • Connect tables and equations when adding, subtracting, or multiplying polynomial functions in real-world and mathematical contexts.  
• Compare the sum and product of two linear functions.  
• Compare characteristics of a combined function with functions that can be used to build it.                                      | A2:F-BF.A.1.b                          | 2              |
| Comparing Characteristics of Functions | • Determine the similarities and differences in characteristics of multiple functions graphically.  
• Determine the similarities and differences in characteristics of multiple functions tabularly.  
• Determine the similarities and differences in characteristics of multiple functions symbolically.                                | A2:F-IF.C.9                            | 1              |
| Modeling with Functions        | • Find the equation of a function that best models a data set.  
• Use function models to solve problems.                                                                                                     | A2:F-IF.B.4, A2:A-CED.A.1, A2:F-IF.C.8.b | 2              |
| Regression Models              | • Determine an exponential, quadratic, or linear model for a given data set using technology.  
• Identify limitations of models in real-world contexts.  
• Use a linear, quadratic, or exponential regression model to make a prediction.  
• Interpret the graph of a regression model in the context of the problem.                                                                 | A2:S-ID.B.6.a, A2:A-CED.A.1            | 1              |
### Lesson Objectives

**Analyzing Functional Relationships**
- Interpret key features of a function represented graphically in terms of a real-world context.
- Interpret key features of a function represented tabularly in terms of a real-world context.
- Graph a function given a verbal description of a relationship.

### Standards

<table>
<thead>
<tr>
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</tr>
</thead>
<tbody>
<tr>
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<td>2</td>
</tr>
<tr>
<td>A2:F-IF.B.6</td>
<td></td>
</tr>
<tr>
<td>A2:F-IF.C.7.b</td>
<td></td>
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</tbody>
</table>

### Performance Task: Production Schemes

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<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>A2:F-IF.B.4</td>
<td>3</td>
</tr>
<tr>
<td>A2:F-IF.B.6</td>
<td></td>
</tr>
<tr>
<td>A2:F-IF.C.7.b</td>
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</tbody>
</table>

### Unit Test

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</table>

### Discussion Questions & Answers

1. A student adds two linear functions, \( f(x) \) and \( g(x) \). The student describes finding the slope of the combined function, \( h(x) \), using a graph as follows:

   - For \( f(x) = 2x + 3 \), the slope is 2, which means for every increase of 2 vertically, there is a corresponding change of +1 horizontally.
   - For \( g(x) = -x + 1 \), the slope is \(-1\), which means for every decrease of 1 vertically, there is a corresponding change of +1 horizontally.
   - When the functions are added, the slope of the sum of the functions must be an increase of \( 2 + (-1) = 1 \) vertically for every corresponding change of \( 1 + 1 = 2 \) horizontally.
   - Therefore, the slope of the sum of the functions is \( \frac{1}{2} \).

   Evaluate the student’s work.

   a. If \( f(x) \) is the linear function \( 2x + 3 \) and \( g(x) \) is the linear function \( -x + 1 \), the slope of \( h(x) \) should be the sum of the slopes of the \( f(x) \) and \( g(x) \). The slope of \( f(x) = 2 \) and the slope of \( g(x) = -1 \). The sum of the slopes is \( 2 - 1 = 1 \). The student added the slopes incorrectly. He or she added \( \frac{2}{1} + \left(\frac{-1}{1}\right) \) by adding both the numerators and the denominators.

2. A linear function, \( f(x) \), is combined by addition with a second function, \( g(x) \). The result is the combined function \( h(x) \), which is a translation of the function \( f(x) \) 5 units down. What must be true of \( g(x) \)? What must be true of the product of \( f(x) \) and \( g(x) \)?

   a. If adding \( f(x) \) and \( g(x) \) results in \( f(x) \) being translated 5 units down, 5 must be subtracted from the \( f(x) \) function to get \( h(x) \). This must mean that \( g(x) = -5 \).

   b. The product of the functions must be linear also, because a linear function, \( f(x) \), is being multiplied by a constant. That constant value does not change the degree of the function, but it does change the associated number. In this case, because the original function, \( f(x) \), is multiplied by \(-5\), the slope of the function that is a result of combining by
multiplication will be five times greater, but in the opposite direction. If the slope of f(x) was positive, it will not be negative, and vice versa.

3. The functions f(x) and g(x) given below describe the heights of two different objects over time, measured in seconds.

\[ f(x) = -x^2 + 10x + 11 \]

<table>
<thead>
<tr>
<th>x</th>
<th>g(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>16</td>
</tr>
<tr>
<td>1</td>
<td>21</td>
</tr>
<tr>
<td>2</td>
<td>24</td>
</tr>
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<td>6</td>
<td>16</td>
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<tr>
<td>7</td>
<td>9</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
</tr>
</tbody>
</table>

Compare the initial heights, maximum heights, time spent in the air, and heights of both objects after 6 seconds.

a. Both initial values can be determined by finding the y-value when x = 0. The initial height of f(x) is 11 and the initial height of the second function is 16. The function g(x) began 5 units higher than f(x).

b. The maximum height based on the equation is the y-coordinate of the maximum. The maximum for the function is (5, 36), making the maximum height for f(x) = 36. The maximum height for g(x) is the greatest y-value on the table: y = 25. The function f(x) traveled 11 units higher than g(x).

c. The time spent in the air is the difference between the initial time, 0, and the time the object reached the ground. For f(x), this is determined by factoring: \(-x + 1)(x - 11)\). This means there are x-intercepts at −1 and 11. The −1 exists before the reasonable domain,
so the x-intercept we are concerned with is 11. The object was in the air for 11 – 0 seconds. For g(x), the object began movement at 0 seconds and had a height of 0 at 8 seconds. This object was in the air for 8 seconds, 3 seconds less than the object associated with f(x).

d. After 6 seconds, the object associated with f(x) was f(6) = −(6)^2 + 10(6) + 11 = 35 feet off the ground. The object associated with g(x), according to the table, was 16 feet off the ground at 6 seconds. Overall, the object associated with f(x) began closer to the ground, but traveled higher and longer than the object associated with g(x).

**Common Misconceptions**

- **Piecewise and Step Functions**
  - Students confuse when a point is included or not included in the domain/range of a function, both graphically and written as an inequality or interval notation.
  - Students may connect points of discontinuity.
  - Students have difficulty understanding contextual variables, such as x – 1 representing “every hour after the first hour.”

- **Combining Functions**
  - When subtracting expressions, students subtract the first term in the expression and then add the remaining terms. Students may not distribute the negative.
  - Students incorrectly combine unlike terms using addition or subtraction.
  - Students might combine functions when the variables represent different things (units, objects, and so on).
  - When combining functions by multiplication, students multiply only like terms rather than distributing.
  - Students compose functions rather than combine functions.
  - Students may believe the domain of the combined function is the sum of the domains of the individual functions.

- **Characteristics of and Modeling with Functions**
  - Students misunderstand or do not know the general shape of various parent functions.
  - Students misunderstand the relationships between successive values in various types of functions.
  - Students confuse (0, y) and (x, 0) and which is the x-intercept and which is the y-intercept.
  - Students confuse the effects h and k have on functions.
  - Students incorrectly believe that any negative r-value means that the relationship between the variables is poor.
  - Students neglect to consider realistic limitations on contextual problems, such as negative or fractional values, or extrapolating beyond what the model could reasonably predict.
**Classroom Challenge**

The sum of two functions, \( f(x) \) and \( g(x) \), is \( 2x^2 + 2x + 3 \).

The product of the same two functions, \( (f \cdot g)(x) \), is \( 2x^3 + zx^2 + 3x \), where \( z \) is an unknown constant.

Give two pairs of expressions that could represent \( f(x) \) and \( g(x) \). Explain both responses, focusing on how you can determine the degrees of the functions \( f(x) \) and \( g(x) \), as well as the values of any coefficients and/or constant terms.

**Possible solution pathway:**

If the functions multiply to be a cubic function, the sum of the highest degrees of both \( f(x) \) and \( g(x) \) has to be 3.

If the functions add to be a quadratic, at least one of the functions \( f(x) \) and/or \( g(x) \) must be quadratic and the other can be quadratic, linear, or a constant.

If one of the functions has to be quadratic, and the sum of the degrees of the functions has to be 3, then the other function has to be linear.

The sum of the quadratic and linear function has to be \( 2x^2 + 2x + 3 \).

\[(ax^2 + bx + c) + (dx + f) = 2x^2 + 2x + 3\]

So, \( a \) must be 2. \( b + d \) must be 2 and \( c + f \) must be 3.

If \( a = 2 \), then the product of the quadratic and linear function begins to look like this:

\[(2x^2 + bx + c)(dx + f) = 2x^3 + zx^2 + 3x\]

Since \( 2x^2 \) times \( x \) is \( 2x^3 \), then \( d \) is 1. If \( d \) is 1, then \( b \) must be 1 because \( b + d = 2 \).

\[(2x^2 + 1x + c)(1x + f) = 2x^3 + zx^2 + 3x\]

\( x \) times \( x \) will be \( x^2 \), so we need an additional \( (z - 1)x^2 \).

There is no constant term in the final product, so \( c \) or \( f \) must be zero. The other value must be 3, since \( c + f \) has to add to 3.

So, the functions \( f(x) \) and \( g(x) \) are either \((2x^2 + x)(x + 3)\) or \((2x^2 + x + 3)(x)\).

**Teacher notes:**

Students who struggle may benefit from being given a value for \( z \) and being asked to find one pair of functions that represents the situation. After finding one solution, the value of \( z \) can be changed so that the second solution can be found. In this case, \( z \) can be 1 or 7.
Students who struggle may also benefit from seeing the added and multiplied functions on a graph.

Students who need additional enrichment can be challenged with functions that are combinations of different types of functions.

UNIT 12: SEQUENCES AND SERIES

Estimated Unit Time: approx. 17 Class Periods

In this unit, students observe patterns and understand how mathematical notation can be used to represent them. Students describe patterns using both explicit and recursive rules, and translate fluently between these forms (MP2). They relate arithmetic sequences to discrete linear functions and geometric sequences to exponential functions. They use, develop, and apply formulas for finding terms of sequences and sums of series. Students recognize arithmetic and geometric sequences and series in the real world and apply their knowledge to solve problems (MP4).

Unit 12 Focus Standards

The following focus standards are intended to guide teachers to be purposeful and strategic in both what to include and what to exclude when teaching this unit. Although each unit emphasizes certain standards, students are exposed to a number of key ideas in each unit, and, as with every rich classroom learning experience, these standards are revisited throughout the course to ensure that students master the concepts with an ever-increasing level of rigor.

In the following table, green highlights indicate major work of the grade, blue highlights indicate supporting work, and yellow highlights indicate additional work.

<table>
<thead>
<tr>
<th>Standard Text</th>
<th>Standard ID</th>
</tr>
</thead>
<tbody>
<tr>
<td>Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.</td>
<td>A2:F-BF.A.2</td>
</tr>
<tr>
<td>Given a graph, a description of a relationship, or two input-output pairs (include reading these from a table), construct linear and exponential functions, including arithmetic and geometric sequences, to solve multistep problems.</td>
<td>A2:F-LE.A.2</td>
</tr>
<tr>
<td>Determine an explicit expression, a recursive process, or steps for calculation from a context.</td>
<td>A2:F-BF.A.1.a</td>
</tr>
<tr>
<td>Apply the formula for the sum of a finite geometric series (when the common ratio is not 1) to solve problems. For example, calculate mortgage payments.</td>
<td>A2:A-SSE.B.4</td>
</tr>
<tr>
<td>Use the properties of exponents to interpret expressions for exponential functions. For example, identify percent rate of change in functions such as $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{12t}$, $y = (1.2)^{6/10}$, and classify them as representing exponential growth or decay.</td>
<td>A2:F-IF.C.8.b</td>
</tr>
</tbody>
</table>
# Unit 12 Pacing Guide

<table>
<thead>
<tr>
<th>Lesson</th>
<th>Objectives</th>
<th>Standards</th>
<th>Number of Days</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Sequences</strong></td>
<td>• Find terms of a sequence from a general formula.</td>
<td>A2:F-BF.A.2</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>• Find the explicit formula of a sequence.</td>
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<tr>
<td></td>
<td>• Describe a pattern represented by a sequence.</td>
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<tr>
<td></td>
<td>• Represent a sequence graphically.</td>
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<tr>
<td></td>
<td><strong>Arithmetic Sequences</strong></td>
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<tr>
<td></td>
<td>• Find the common difference of an arithmetic sequence.</td>
<td>A2:F-BF.A.2</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>• Determine if a sequence is arithmetic.</td>
<td>A2:F-BF.A.2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Apply the formula of an arithmetic sequence.</td>
<td>A2:F-LE.A.2</td>
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</tr>
<tr>
<td></td>
<td>• Find the terms of an arithmetic sequence.</td>
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<tr>
<td></td>
<td><strong>Geometric Sequences</strong></td>
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<tr>
<td></td>
<td>• Find the common ratio of a geometric sequence.</td>
<td>A2:F-BF.A.2</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>• Determine if a sequence is geometric.</td>
<td>A2:F-LE.A.2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Apply the formula of a geometric sequence.</td>
<td></td>
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<tr>
<td></td>
<td>• Find terms of a geometric sequence.</td>
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<tr>
<td></td>
<td><strong>Arithmetic Series</strong></td>
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<tr>
<td></td>
<td>• Solve problems using the formula for the sum for an arithmetic series.</td>
<td>A2:F-BF.A.1.a</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td><strong>Geometric Series</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Find sums of finite and infinite geometric series.</td>
<td>A2:A-SSE.B.4</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>• Apply geometric series to solve mathematical and real-world problems.</td>
<td>A2:F-BF.A.1.a</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Find terms of a geometric sequence.</td>
<td>A2:F-IF.C.8.b</td>
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<tr>
<td></td>
<td><strong>Finite Geometric Series</strong></td>
<td></td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>• Solve problems using the formula for the sum of a finite geometric series.</td>
<td>A2:A-SSE.B.4</td>
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<tr>
<td></td>
<td><strong>Recursive Formulas</strong></td>
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<tr>
<td></td>
<td>• Write the first $n$ terms of a recursive function given a formula and a term.</td>
<td>A2:F-BF.A.1.a</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>• Write a rule for a recursively defined function.</td>
<td>A2:F-BF.A.2</td>
<td></td>
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<tr>
<td></td>
<td><strong>Modeling with Sequences and Series</strong></td>
<td></td>
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<tr>
<td></td>
<td>• Determine if a sequence or series is arithmetic or geometric.</td>
<td>A2:F-BF.A.1.a</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>• Solve real-world problems involving sequences.</td>
<td>A2:F-BF.A.2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Solve real-world problems involving series.</td>
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<tr>
<td><strong>Unit Test</strong></td>
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<td>1</td>
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</tbody>
</table>
Discussion Questions & Answers

1. In the arithmetic sequence \(a, b, c, \ldots\), how must \(b\) be related to \(a\) and \(c\)?
   
   a. The value of \(a\) plus the common difference, \(d\), must be \(b\). The value of \(b\) plus the common difference, \(d\), must be \(c\). This means \(a + d = b\) and \(b + d = c\). We can solve each equation for \(d\):
      
      \[
      a + d = b \Rightarrow d = b - a \\
      b + d = c \Rightarrow d = c - b
      \]
   
      Then because the equations are both equal to \(d\), we can set them equal to each other:
      
      \[
      b - a = c - b
      \]
   
      Next, we combine like terms:
      
      \[
      2b - a = c
      \]
   
      Finally, we isolate \(b\):
      
      \[
      2b = c + a \Rightarrow b = \frac{c + a}{2}
      \]

      The middle term, \(b\), is the arithmetic average of the term immediately preceding and immediately following.

2. Explain why the average of all the numbers in a finite arithmetic sequence is equal to the average of the first and last terms.
   
   a. Starting with a middle number in the sequence, we can have a value \(n\). The values immediately preceding and immediately following \(n\) are \(n - d\) and \(n + d\), where \(d\) is the common difference of the arithmetic sequence. The sum of these three terms is \(n + (n - d) + (n + d) = 3n\). Divide by the number of terms to get the average: \(3n\) divided by \(3\) is \(n\). This is equal to our initial average, or middle value. The value is \(x\) number of common differences away, in both directions. This can be represented by \(n - xd\) and \(n + xd\). The average of these numbers, any number of common differences away from \(n\) is
      
      \[
      \frac{n - xd + n + xd}{2}
      \]
   
      which equals \(n\). Because \(x\) represents any number of common differences away, this can include our initial and ending values. The average of the whole set of numbers would include \((n - xd) + \ldots + (n - d) + n + (n + d) + \ldots + (n + xd)\). Notice the symmetry in the set and that each symmetric pair averages to the initial average. Because the first and last terms are symmetric pairs in this sequence, they will have the same average as the average of the set.

3. Explain how to use a convergent geometric series to write \(0.\overline{7}\) as a fraction.
   
   a. The formula for the sum of a convergent series is
      
      \[
      S = \frac{a_1}{1 - r}
      \]
   
      This series can be represented by the fractions:
      
      \[
      \frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \ldots
      \]
   
      This means \(a_1 = \frac{7}{10}\) and \(r = \frac{1}{10}\) because we are multiplying by \(\frac{1}{10}\) to get from one number to the next consecutive number.
   
      So:
      
      \[
      S = \frac{\frac{7}{10}}{1 - \frac{1}{10}} = \frac{\frac{7}{10}}{\frac{9}{10}} = \frac{7}{9}
      \]
Common Misconceptions

- Sequences
  - Students generate a rule for a sequence that only applies to a portion of the data set.
  - In arithmetic sequences, students confuse negative and positive common differences.
  - Students do not understand the relationship between the slope of a linear function and the common difference.
  - Students incorrectly use a negative common ratio to show that the sequence is constantly decreasing.
  - Students confuse recursive and explicit formulas.

- Arithmetic and Geometric Series
  - Students memorize formulas without internalizing the meanings of the variables and the goal of using the formulas.
  - Student assumes the sum of an infinite geometric series cannot be calculated, or that all geometric series diverge.
  - Students may consider subscripts as an operation rather than a label.

Classroom Challenge

Give two distinct examples of convergent geometric series that have different $r$ values, but the same sum. Compare the $r$ values, then compare the initial values. What do you notice?

Now, we will see if the observation you made about your two series applies to any two convergent geometric series with different $r$ values but the same sum.

Consider any two convergent geometric series that have the same sum. The first geometric series has an initial value $a$ and a common ratio $r_1$. The second geometric series has an initial value $b$ and a common ratio $r_2$. If $r_1 > r_2$, how must the initial values, $a$ and $b$, compare? Explain your answer.

Possible solution pathway:

For the first series, I used $3, 2, \frac{4}{3}, \ldots$ so I have an $a$ value of 3 and an $r$ value of $\frac{2}{3}$.

My sum is: $\frac{a}{1-r} = \frac{3}{1-\frac{2}{3}} = \frac{3}{\frac{1}{3}} = 9$.

Then, I used a different ratio, $\frac{1}{2}$, with the sum of 9.

$\frac{a}{1-r} = \frac{a}{1-\frac{1}{2}} = \frac{a}{\frac{1}{2}} = 2a = 9$, so $a = 4.5$.

My first series is $3, 2, \frac{4}{3}, \ldots$
My second series is $4.5, 2.25, 1.125, \ldots$
They both have a sum of 9. My first series has an $a$ value of 3 which is smaller than 4.5, the $a$ value of my second series. My first series has an $r$ value of $\frac{2}{3}$, which is larger than $\frac{1}{2}$, the $r$ value of my second series. I notice that if two series have the same sum, one series has a big common ratio and a small initial value while the other series has a small common ratio and a big initial value.

If the sums of the series are the same, then

$$\frac{a}{1-r_1} = \frac{b}{1-r_2}$$

If $r_1 > r_2$, then the denominator of the second fraction will be larger than the denominator of the first fraction, since both are subtracted from 1. In order for the fractions to be equal to each other, the numerator of the second fraction, $b$, will have to be larger than the numerator of the first fraction, $a$. $a < b$.

**Teacher notes:**

Students who struggle may benefit from being asked to use distinct values for $r_1$ and $r_2$ in order to make their generalization about $a$ and $b$.

Students who need enrichment may benefit from being asked to write an expression or an equation that can be used to determine the factor by which the original initial value, $a$, must be multiplied (in terms of $r_1$ and $r_2$) in order to find the initial value of the new series (assuming the sums of the convergent geometric series are the same).

**UNIT 13: STATISTICS AND PROBABILITY**

_Estimated Unit Time: approx. 13 Class Periods_

In this unit, students learn about different ways to collect data, including sample surveys, experiments, and observational studies. From there, they learn how to calculate various descriptive statistics of a sample, such as standard deviation. Students calculate confidence intervals to make an inference about a population parameter from a sample statistic and find the margin of error. They learn about probability distributions and use them to solve real-world problems (MP4). Students determine statistical significance of an experiment outcome by performing hypothesis testing.

**Unit 13 Focus Standards**

The following focus standards are intended to guide teachers to be purposeful and strategic in both what to include and what to exclude when teaching this unit. Although each unit emphasizes certain standards, students are exposed to a number of key ideas in each unit, and, as with every rich classroom learning experience, these standards are revisited throughout the course to ensure that students master the concepts with an ever-increasing level of rigor.

In the following table, green highlights indicate major work of the grade, blue highlights indicate supporting work, and yellow highlights indicate additional work.
### Standard Text

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</tr>
</thead>
<tbody>
<tr>
<td>Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each.</td>
<td>A2:S-IC.B.3</td>
</tr>
<tr>
<td>Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve.</td>
<td>A2:S-ID.A.4</td>
</tr>
<tr>
<td>Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling.</td>
<td>A2:S-IC.B.4</td>
</tr>
<tr>
<td>Understand statistics as a process for making inferences to be made about population parameters based on a random sample from that population.</td>
<td>A2:S-IC.A.1</td>
</tr>
<tr>
<td>Evaluate reports based on data.</td>
<td>A2:S-IC.B.6</td>
</tr>
<tr>
<td>Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation. For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model?</td>
<td>A2:S-IC.A.2</td>
</tr>
<tr>
<td>Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant.</td>
<td>A2:S-IC.B.5</td>
</tr>
</tbody>
</table>

### Unit 13 Pacing Guide

<table>
<thead>
<tr>
<th>Lesson</th>
<th>Objectives</th>
<th>Standards</th>
<th>Number of Days</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Designing a Study</strong></td>
<td>• Classify study types.</td>
<td>A2:S-IC.B.3</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>• Classify sampling methods.</td>
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<tr>
<td></td>
<td>• Determine if a sample is biased.</td>
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<tr>
<td></td>
<td>• Analyze study types and sampling methods.</td>
<td></td>
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<tr>
<td><strong>Representing Data</strong></td>
<td>• Describe a data set using measures of central tendency and range.</td>
<td>A2:S-ID.A.4</td>
<td>1</td>
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<tr>
<td></td>
<td>• Determine if a representation of data is misleading.</td>
<td></td>
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<tr>
<td><strong>Standard Deviation</strong></td>
<td>• Calculate variance and standard deviation of a sample or population.</td>
<td>A2:S-IC.B.4</td>
<td>1.5</td>
</tr>
<tr>
<td></td>
<td>• Interpret standard deviation as it pertains to the spread of a graph.</td>
<td></td>
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<tr>
<td></td>
<td>• Determine if a value is within a given z-score.</td>
<td></td>
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<tr>
<td>Lesson</td>
<td>Objectives</td>
<td>Standards</td>
<td>Number of Days</td>
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</tbody>
</table>
| Properties of Probability Distributions | - Identify properties of a probability distribution.  
- Create probability distributions from a data set.  
- Solve problems using probability distributions. | A2:S-IC.A.1     | 1              |
| Expected Value              | - Calculate expected values.  
- Use expected values to make decisions. | A2:S-ID.A.4     | 1              |
| Binomial Distribution       | - Identify a binomial experiment.  
- Identify the probability of success, probability of failure, and number of trials for a binomial experiment.  
- Calculate binomial probabilities. | A2:S-IC.B.6     | 1.5            |
| Introduction to Normal Distributions | - Describe normal distributions using the mean and standard deviation.  
- Apply the z-score formula to solve problems.  
- Solve problems using the empirical rule. | A2:S-ID.A.4     | 1              |
| Applications with Standard Normal Distribution | - Solve problems using the standard normal table. | A2:S-ID.A.4     | 1              |
| Statistical Inferences      | - Make inferences about a population from a sample. | A2:S-IC.B.4     | 1.5            |
|                             |                                                                             | A2:S-IC.A.2     |                 |
|                             |                                                                             | A2:S-IC.A.1     |                 |
| Hypothesis Testing          | - Perform hypothesis tests on normally distributed data.  
- Determine if a result is statistically significant. | A2:S-IC.B.5     | 1.5            |
| Unit Test                   |                                                                             |                 | 1              |

**Discussion Questions & Answers**

1. What are the steps for calculating a z-score?
   a. A z-score tells us how far a data point is from the mean. To calculate a z-score, we need the standard deviation and mean of a data set. First, calculate the mean of the data set by adding all the values and dividing by the number of values. Next, calculate the variance by taking each data point, subtracting the mean, and squaring. Find the average of these values. Then, to find the standard deviation, take the square root of the variance. Finally, to find the z-score, subtract the mean from the value in question. Divide that difference by the standard deviation. This will give you the z-score, which
tells you how many standard deviations the value in question is away from the mean of the data set.

2. A student flips a fair coin 5 times. He flips ‘heads’ 4 times, and considers flipping ‘heads’ a success. Explain why flipping ‘heads’ 1 time has the same probability.
   a. Using the formula to determine the probability of success for 4 heads, we have
      \[ \binom{5}{4} \left( \frac{1}{2} \right)^4 \left( \frac{1}{2} \right)^1 = 5 \left( \frac{1}{16} \right) \left( \frac{1}{2} \right) = \frac{5}{32}. \]
      Using the same formula to determine the probability of success for 1 head, we have
      \[ \binom{5}{1} \left( \frac{1}{2} \right)^1 \left( \frac{1}{2} \right)^4 = 5 \left( \frac{1}{2} \right) \left( \frac{1}{16} \right) = \frac{5}{32}. \]
      The powers to which our equal probability is raised are just switched. This switch does not matter when the results are multiplied; \( \binom{5}{4} \) and \( \binom{5}{1} \) are equivalent.
      
      You can think about it this way: The probability of having 4 heads out of 5 is the same as having 4 tails out of 5, since the probabilities of flipping heads and flipping tails are equal. Since 4 tails out of 5 is the same as 1 head out of 5, 1 head out of 5 has the same probability as 4 heads out of 5.

3. The margin of error for a sample survey is large. What can be done to decrease the margin of error? (Note: Assume the standard deviation cannot arbitrarily be changed without supporting changes to the data set.)
   a. The margin of error is determined by the formula
      \[ ME = \frac{z^*\sigma}{\sqrt{n}}. \]
      To decrease the margin of error, the \( z^* \) could be decreased. This means that instead of using, for example, a 95% confidence level, we might use a 90% confidence level. The sample size could also be increased.

Common Misconceptions

- Developing a Study
  o Students confuse simple random sampling with systematic random sampling, and stratified sampling with cluster sampling.
  o Students may not see bias when the bias is in their favor or creates the expected result.

- Representing Data
  o Students read the data off a frequency table as a single value instead of the value times the frequency.
  o Students may associate direction of skew with the majority of the data points instead of the location of the tail.
  o Students find the median on a histogram by finding the middle number on the \( x \)-axis instead of the middle number using the heights of the bars.
  o Students might choose to include or not include the median when finding the quartiles.
• **Standard Deviation**
  o Students might memorize formulas instead of understanding what is being calculated at each step.
  o Students may believe that a z-score tells the numerical distance from the mean instead of the number of standard deviations from the mean.

• **Probability Distributions**
  o Students create a table or graph based on frequency instead of probability of an event occurring.
  o Students confuse the relationship between the mean and median in skewed data sets.
  o Students inaccurately calculate the probability of success across multiple events.
  o Students assume that a “normal” distribution is the same as uniform distribution (i.e., each data point occurs the same number of times).
  o Students may apply a formula without understanding the variables and/or outcome in the context of the problem.

• **Expected Value**
  o Students may decide based on a gut feeling, risk aversion, etc., rather than expected value.
  o Students calculate the probability of independent events using an operation other than addition.

• **Statistical Inferences**
  o Instead of understanding what each formula is used to compute, students memorize the formulas (and common values, such as z* associated with 95% confidence levels) and have a hard time knowing where to start to achieve their goal.
  o Students may make inferences based on faulty, biased, or non-representative samples.
  o Students draw conclusions not supported, either mathematically or contextually, by the data.

• **Hypothesis Testing**
  o Students might decide the accuracy of a statement based on strict equality instead of considering statistical variation.

**Classroom Challenge**
Create three different probability games. Each game must use a different score-generating tool (such as a spinner). In each game, the player must have the chance to “win” and the chance to “lose.” Be sure to describe the score-generating tool and what is considered a “win.”

Game A must be a fair game.
Game B must be a game whose expected outcome favors the player.
Game C must be a game whose expected outcome is non-favorable to the player.

For each game, describe the expected value based on the context you describe.
Possible Solution Pathway:

Game A: A spinner containing 2 sections, each taking up half of the circle, is spun. One section is red and one section is blue. If red is spun, it is considered a win and the player scores 2 points. If blue is spun, the player loses 2 points.

The expected value in Game A is 0. There is a 0.50 probability that red will be spun, resulting in a gain of 2 points: $0.50 \times 2 = 1$. There is a 0.50 probability that blue will be spun, resulting in a loss of 2 points: $0.50 \times -2 = -1$. The sum of 1 and -1 is 0. The game is fair.

Game B: A number cube is rolled. If a factor of 6 is rolled, the player wins 2 points. If the number rolled is not a factor of 6, the player loses 2 points.

The expected value in Game B is $\frac{2}{3}$. This is favorable to the player. There is a $\frac{2}{3}$ probability that a factor of 6 will be rolled (1, 2, 3, 6), resulting in a gain of 2 points: $\frac{2}{3} \times 2 = \frac{4}{3}$. There is a $\frac{1}{3}$ probability that a number will be rolled that is not a factor of 6 (4 and 5), resulting in a loss of 2 points: $\frac{1}{3} \times (-2) = -\frac{2}{3}$. The sum of these values is $\frac{2}{3}$.

Game C: Two fair coins are flipped. A win is considered 2 of the same outcome (either heads/heads or tails/tails). For each win, a player receives 5 points. For each loss (heads/tails), the player loses 7 points.

The expected value in Game C is -1. There are 4 possible outcomes, HH, HT, TH, and TT. HH and TT are “wins,” so there is a probability of $\frac{1}{2}$ that the player will win. $\frac{1}{2} \times 5 = 2.5$. There is a probability of $\frac{1}{2}$ that the play will flip HT or TH and lose. $\frac{1}{2} \times (-7) = -3.5$. The sum of these values is -1, making the game not favorable for the player.

Teacher notes:

Students who struggle may benefit from being permitted to use the same probability-generating device (spinner, coins, etc.) for each game. These students may also benefit from reminders that they can manipulate the probability of a “win” or “loss” by adjusting their probability-generating tool, and that they can adjust the value of a “win” or a “loss.”

Students who would benefit from extensions may be asked to use compound probabilities when determining what constitutes a “win” or a “loss.”

**TIPS ON EFFECTIVE DISCUSSIONS**

Edgenuity courseware supports students in using mathematical language precisely (MP6), and constructing viable arguments to justify their reasoning (MP3) by including discussion questions for students to complete in a classroom or virtual discussion room setting.
• Make expectations clear. How many times should students post in discussions? Are they required to respond to all questions or just one in each unit? You may also wish to require that students respond to at least one other student’s post in addition to answering the original question themselves.

• Share appropriate rules for online discussions with students. Remind them that online discussions don’t convey tone as well as face-to-face discussions, and they should be careful to write things that cannot be misinterpreted (e.g., avoid sarcasm). Likewise, let students know that they should never post anything in a discussion forum that they would not say in a face-to-face discussion.

• Encourage students to ask follow-up questions in the forum. Discuss with students what makes an effective follow-up question. For example, questions should elicit elaboration, as opposed to having a single correct answer.

• Let students know that there is no shame in changing one’s views in response to new information posted by others. In fact, that is part of what discussions are about.

• If you facilitate online discussions, post discussion questions at the start of the unit so students can return to them multiple times. Students are often motivated to contribute more when they see the contributions of other students over time. Once you have posted questions, send students an email to let them know that forums are open.

• If you facilitate face-to-face discussions, make clear to students what they should come prepared to discuss. You may wish to provide the discussion questions in advance of the discussion. When discussion is happening in real time, this allows less-confident students to feel more prepared and have evidence to support their ideas.
## Course Vocabulary List

All of the words below are taught to students in the course of instruction. Students have access to definitions in their lesson glossaries and can look up any word at any time within instruction and assignments.

<table>
<thead>
<tr>
<th>Term</th>
<th>Term</th>
<th>Term</th>
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</thead>
<tbody>
<tr>
<td>addition property of inequality</td>
<td>common logarithm</td>
<td>direct variation</td>
</tr>
<tr>
<td>alternative hypothesis (Ha)</td>
<td>common ratio</td>
<td>discriminant</td>
</tr>
<tr>
<td>amplitude</td>
<td>complex conjugate</td>
<td>divergent series</td>
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<tr>
<td>arithmetic sequence</td>
<td>complex conjugate theorem</td>
<td>domain</td>
</tr>
<tr>
<td>arithmetic series</td>
<td>complex fraction</td>
<td>end behavior</td>
</tr>
<tr>
<td>axis (or line) of symmetry</td>
<td>composition of functions</td>
<td>evaluate</td>
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<tr>
<td>base</td>
<td>compound inequality</td>
<td>expected value</td>
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<tr>
<td>bias</td>
<td>confidence interval</td>
<td>experiment</td>
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<tr>
<td>binomial distribution</td>
<td>confidence level</td>
<td>explicit formula</td>
</tr>
<tr>
<td>binomial expansion</td>
<td>consistent system</td>
<td>exponential decay</td>
</tr>
<tr>
<td>binomial experiment</td>
<td>constant of variation</td>
<td>exponential equation</td>
</tr>
<tr>
<td>binomial theorem</td>
<td>continuous function</td>
<td>exponential function</td>
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<tr>
<td>binomial trial</td>
<td>convenience sampling</td>
<td>exponential growth</td>
</tr>
<tr>
<td>ceiling function</td>
<td>convergent series</td>
<td>exponentiate (an expression)</td>
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<tr>
<td>census</td>
<td>cosecant</td>
<td>extraneous solution</td>
</tr>
<tr>
<td>central angle</td>
<td>cotangent</td>
<td>extrapolation</td>
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<tr>
<td>change of base formula</td>
<td>cotermination side</td>
<td>fair game</td>
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<td>cluster sampling</td>
<td>critical region</td>
<td>finite sequence</td>
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<tr>
<td>coefficient of determination</td>
<td>cube root</td>
<td>frequency</td>
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<tr>
<td>combined function</td>
<td>dependent system</td>
<td>function</td>
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<tr>
<td>common difference (d)</td>
<td>dependent variable</td>
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<tr>
<td><strong>fundamental theorem of algebra (and corollary)</strong></td>
<td><strong>mean of a probability distribution</strong></td>
<td><strong>probability distribution</strong></td>
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<tr>
<td><strong>geometric sequence</strong></td>
<td><strong>minimum</strong></td>
<td><strong>product property of logarithms</strong></td>
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<tr>
<td><strong>geometric series</strong></td>
<td><strong>monomial function</strong></td>
<td><strong>product property of roots</strong></td>
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<tr>
<td><strong>greatest integer function</strong></td>
<td><strong>multiplication property of inequality</strong></td>
<td><strong>quadrantal angle</strong></td>
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<tr>
<td><strong>horizontal asymptote</strong></td>
<td><strong>multiplicity of a zero</strong></td>
<td><strong>quadratic formula</strong></td>
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<tr>
<td><strong>hypothesis test</strong></td>
<td><strong>natural logarithm</strong></td>
<td><strong>quadratic function</strong></td>
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<tr>
<td><strong>imaginary number</strong></td>
<td><strong>normal distribution</strong></td>
<td><strong>quotient property of logarithms</strong></td>
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<tr>
<td><strong>inconsistent system</strong></td>
<td><strong>null hypothesis (H0)</strong></td>
<td><strong>quotient property of roots</strong></td>
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<tr>
<td><strong>independent events</strong></td>
<td><strong>observational study</strong></td>
<td><strong>radian</strong></td>
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<tr>
<td><strong>independent system</strong></td>
<td><strong>one-to-one</strong></td>
<td><strong>radical equations</strong></td>
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<tr>
<td><strong>independent variable</strong></td>
<td><strong>parabola</strong></td>
<td><strong>radicand</strong></td>
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<tr>
<td><strong>infinite sequence</strong></td>
<td><strong>parameter</strong></td>
<td><strong>random variable</strong></td>
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<tr>
<td><strong>interpolation</strong></td>
<td><strong>parent function</strong></td>
<td><strong>range</strong></td>
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<tr>
<td><strong>inverse of a function</strong></td>
<td><strong>perfect nth power</strong></td>
<td><strong>rate of change</strong></td>
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<tr>
<td><strong>inverse variation</strong></td>
<td><strong>perfect-square trinomial</strong></td>
<td><strong>rational equation</strong></td>
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<tr>
<td><strong>irrational root theorem</strong></td>
<td><strong>period</strong></td>
<td><strong>rational expression</strong></td>
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<tr>
<td><strong>isolate</strong></td>
<td><strong>phase shift</strong></td>
<td><strong>rational function</strong></td>
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<tr>
<td><strong>law of large numbers</strong></td>
<td><strong>piecewise defined function</strong></td>
<td><strong>rational root theorem</strong></td>
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<tr>
<td><strong>like radicals</strong></td>
<td><strong>point-slope form</strong></td>
<td><strong>rationalize the denominator</strong></td>
</tr>
<tr>
<td><strong>linear function</strong></td>
<td><strong>power property of logarithms</strong></td>
<td><strong>recursive formula</strong></td>
</tr>
<tr>
<td><strong>local maximum</strong></td>
<td><strong>predict</strong></td>
<td><strong>reference angle</strong></td>
</tr>
<tr>
<td><strong>local minimum</strong></td>
<td><strong>prime factorization</strong></td>
<td><strong>reflection</strong></td>
</tr>
<tr>
<td><strong>logarithm</strong></td>
<td><strong>prime polynomial</strong></td>
<td><strong>regression</strong></td>
</tr>
<tr>
<td><strong>logarithmic function</strong></td>
<td><strong>probability</strong></td>
<td><strong>relation</strong></td>
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<tr>
<td><strong>maximum</strong></td>
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</tbody>
</table>
remainder theorem | standard deviation | transformation
---|---|---
sample space | standard form of a linear equation | transitive property of inequality
sample survey | standard normal distribution | translation
scatterplot | standard position | turning point
secant | standardize a random variable | unit circle
sequence | statistic | variance
significance level (theta) | statistical inference | vertex
simple random sampling | stratified sampling | vertex form of a quadratic equation
simulation | substitute | vertical asymptote
slope-intercept form | symmetric | voluntary sampling
solution of a system of equations | synthetic division | x-intercept
solution to a system of equations | system of equations | y-intercept
square root | systematic random sampling | zero of a function
square root function | term | zero product property
square root property of equality | terminal side | z-score

**Edgenuity Algebra II Interactive Tools**

Students use a variety of powerful interactive instructional tools to help them build content knowledge and essential skills, support them in learning procedures, and facilitate the exploration of new or challenging concepts throughout Edgenuity Algebra II.

**UNIT 1**

**Linear Functions**

- An interactive graph allows students to explore how slope and y-intercept change a line.
UNIT 2

Relations and Functions
- An interactive tool supports students in determining output values of plotted values, as well as using the vertical line test to determine if the lines are a function.

Function Inverses
- An interactive graph lets students explore inverse functions.

Rate of Change
- An interactive graph allows students to change the value of $k$ in $y = kx$ to explore how that affects the slope of the line.
- A graph allows students to explore a changing average rate of change.

UNIT 3

Complex Numbers
- An interactive graph allows students to investigate the definition of absolute value.

Transformations of Quadratic Functions
- An interactive graph explores a vertical translation of $y = x^2$.
- An interactive graph explores a horizontal translation of $y = x^2$.
- An interactive graph explores the effect of $a$ in the equation $y = ax^2$.

The Square Root Function
- An interactive graph allows students to plot a square root function.

UNIT 4

Solving Linear Systems by Substitution
- An interactive assignment uses linear systems to uncover dinosaur bones.

UNIT 5

N/A

UNIT 6

Graphing Polynomial Functions
- An interactive assignment using polynomial functions to create a basic model of a roller coaster.
UNIT 7

N/A

UNIT 8

Graphing Radical Functions
- An interactive graph assists students in finding the domain and range of square root functions.
- An interactive graph assists students in exploring dilations of square root functions.

Rational Exponents
- An interactive assignment uses rational exponents to explore the solar system.

UNIT 9

Exponential Growth Functions
- An interactive graph allows students to plot an exponential growth function.

Graphing Exponential Functions
- An interactive graph supports students in exploring transformations of $y = 2^x$.

Modeling with Exponential and Logarithmic Equations
- An interactive assignment uses exponential models to explore the unusual population growth of water lilies.

UNIT 10

Graphing Sine and Cosine
- An interactive graph supports students in exploring amplitude of the sine function.

Changes in Period and Phase Shift of Sine and Cosine Functions
- An interactive graph supports students in exploring change of frequency in the sine function.
- An interactive graph supports students in exploring a phase shift of the cosine function.

Modeling with Periodic Functions
- An interactive assignment explores using a periodic function to model a rope swing.

UNIT 11

Regression Models
• An interactive graph supports students in selecting a type of function to model data.

UNIT 12
N/A

UNIT 13

Expected Value
• Interactive dice simulators roll a dice and allow students to explore expected value.

Binomial Distribution
• Interactive coin and spinner simulators allow students to explore binomial distribution.

COURSE CUSTOMIZATION

Edgenuity is pleased to provide an extensive course customization toolset, which allows permissioned educators and district administrators to create truly customized courses that ensure that our courses can meet the demands of the most rigorous classroom or provide targeted assistance for struggling students.

Edgenuity allows teachers to add additional content two ways:

1. Create a brand-new course: Using an existing course as a template, you can remove content; add lessons from the Edgenuity lesson library; create your own activities; and reorder units, lessons, and activities.

2. Customize a course for an individual student: Change an individual enrollment to remove content; add lessons; add individualized activities; and reorder units, lessons, and activities.

Below you will find a quick-start guide for adding lessons from a different course or from our lesson library.
In addition to adding lessons from another course or from our lesson library, Edgenuity teachers can insert their own custom writing prompts, activities, and projects.
How do I Create Project or Writing Prompt Activities?

Navigate to where you want to add the new activity, and select the lesson by clicking on the lesson title.

Click the Add Activity button, then select Writing Prompt or Project.

If you previously created new activities, they will display here. Click the activity name to preview the activity instructions.

Click the green plus sign to insert an activity into the lesson.

The activity will be inserted at the top of the unit. You can move the activity to another location in the lesson.

If you are creating a new Writing Prompt, specify the name, description, prompt, grade weight category, and optionally, keywords for scoring, sample answer, and scoring guidance.

If you are creating a new Project, specify the name, description, type, and grade weight category, and provide student resources by entering hyperlinks to web sites or uploading files.

NOTES

- Accepted file types are: .ppt, .pptx, .xls, .xlsx, .doc, .docx, .zip, .pdf, .acdb, .msg.

- Links you create won’t go through the Edgenuity Emissary (Proxy). This means that your IT department will need to ensure that the link is whitelisted or otherwise allowed to be accessed. It also means that items blocked by the Edgenuity proxy may be visible on the sites you link to. In addition, the Edgenuity tools to highlight, translate, read aloud, or add a sticky note will not be present on the site you link to.

- You add activities to courses with no enrollments or on individual student’s courses. It is not possible to add activities to in-flight courses that have enrollments.